PhD seminar @ PhD Away Days 2019

Université du Bordeaux, Bordeaux, France

What is martingale?

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A Digression

A digression - conditional expectation

In the following, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Let A and B be two events. Then the conditional probability is given by

$$\mathbb{P}(A \mid B) = ?.$$

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$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

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Let $A = \{X = x\}$ and $B = \{Y = y\}$ be two events. Then the conditional probability is given by

$$\mathbb{P}(X = x \mid Y = y) = \frac{\mathbb{P}\left(\{X = x\} \cap \{Y = y\}\right)}{\mathbb{P}(Y = y)}.$$

The **expectation** of X is

$$\mathbb{E}(X) = \sum_x x\,\mathbb{P}(X = x).$$

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The **conditional expectation** of X given $Y = y$ is

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Example

We flip a coin twice. $X \triangleq \sharp\{\text{heads in 1 flip}\}$

$Y \triangleq \sharp\{\text{heads in 2 flips}\}$

Get for all possible values of Y:

$$\mathbb{E}(X \mid Y = 0) = 0, \qquad \mathbb{E}(X \mid Y = 1) = \frac{1}{2}, \qquad \mathbb{E}(X \mid Y = 2) = 1.$$

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Martingales

Setting

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$\mathscr{F}_n \subset \mathscr{F}$ sub σ-algebras, $\mathscr{F}_n \uparrow$ events observable up to time n

We gamble with fixed bet and winning 1.

$\xi_j \in L^1$ iid	winning in the n^th draw
$S_n = \xi_1 + \ldots + \xi_n$	overall winning after n draws
$S_0 = 0$	
$\mathscr{F}_n = \sigma(\xi_1, \ldots, \xi_n)$	information about all draws $1, 2, \ldots, n$
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Let's vary the game and allow variable bets, i.e.

$$e_{n+1} = e_{n+1}(\xi_1, \ldots, \xi_n) \geqslant 0 \qquad \text{bet in the } n+1^{\text{th}} \text{ draw.}$$

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Question What is a martingale?

(Heuristic) Answer A martingale is the generalisation of the concept of a (fair) game.

Definition

Let I be an index set (ordered) and $(\mathscr{F}_t)_{t\in I}$ a filtration. A stochastic process $(X_t)_{t\geqslant 0}$ satisfying, $\forall t \geqslant 0$,

(i) X_t is $\mathscr{F}_t$-adapted, i.e. $X_t \in \mathscr{F}_t$ measurable;

(ii) $X_t \in \mathrm{L}^1$, i.e. $\mathbb{E}\,|X| < \infty$;

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Example (Doubling strategy)

Let $X_1, \ldots, X_n$ be iid random variables such that

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

We think of X_i describing the result of a coin flipping game:

- win 1€ if coin comes up heads
- lose 1€ if coin comes up tails

Consider **doubling strategy**, i.e. keep doubling the bet until we eventually win. once we win, we stop and our initial bet is 1€.

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W_n only takes two possible values, $W_n \in \{1, -2^n + 1\}$ $(n \in \mathbb{N})$. **Why?**

(1) Suppose we win for the first time on the n^{th} bet. Then

$$\begin{aligned} W_n &= -(1 + 2 + \dots + 2^{n-2}) + 2^{n-1} \\ &= -(2^{n-1} - 1) + 2^{n-1} \\ &= 1 \end{aligned}$$

(2) If we have not yet won after n bets, then

$$\begin{aligned} W_n &= -(1 + 2 + \dots + 2^{n-1}) \\ &= -2^n + 1. \end{aligned}$$

Show $\mathbb{E}(W_{n+1} \mid W_n) = W_n \overset{\text{iteration}}{\Longrightarrow} \mathbb{E}(W_{n+m} \mid W_n) = W_n$

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(1) $W_n = 1$: then $\mathbb{P}(W_{n+1} = 1 \mid W_n = 1) = 1$, so

$$\mathbb{E}(W_{n+1} \mid W_n = 1) = 1 = W_n.$$

(2) $W_n = -2^n + 1$: bet 2^n on $(n+1)^{\text{th}}$ toss, so $W_{n+1} \in \{1, -2^{n+1} + 1\}$. Clearly,

$$\mathbb{P}\left(W_{n+1} = 1 \mid W_n = -2^n + 1\right) = \frac{1}{2}$$

$$\mathbb{P}\left(W_{n+1} = -2^{n+1} + 1 \mid W_n = -2^n + 1\right) = \frac{1}{2}$$

so that

$$\begin{aligned}\mathbb{E}\left(W_{n+1} \mid W_n = -2^n + 1\right) &= 1 \cdot \frac{1}{2} + \left(-2^{n+1} + 1\right)\frac{1}{2} \\ &= -2^n + 1 = W_n.\end{aligned}$$

Applications

- Doob's optional stopping theorem $\longrightarrow$ on average, nothing can be winninged by stopping play based on the information obtainable w/o looking into the future
- most essential continuous time process, Brownian Motion, is a martingale (zero drift stochastic process)
- martingales are essential to stochastic integration: $\mathbb{E}B_t^2 = t$, Doob-Meyer decomposition.
- solve PDEs by stochastic/martingale methods
- stochastic differential geometry, stochastic calculus on manifolds (and beyond)
- finance: martingality of an asset is equivalent to not being able to conduct arbitrage through trades in that asset
- ...

Most prominent example

Example (Brownian motion)

Let (M, g) be a smooth Riemannian manifold, $B = (B_t)$ Brownian motion on M and $\mathbf{A} = \frac{1}{2}\Delta_M$ with Δ_M the Laplace-Beltrami operator on M, i.e. $\Delta = \sum_{i=1}^n \partial_i^2$ on $\mathbb{R}^n$. Then,

$$\forall f \in C_c^\infty(\mathbb{R}^n), t \geqslant 0 : \quad f(B_t) - f(B_0) - \int_0^t \frac{1}{2}\Delta f(B_s)\mathrm{d}s$$

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Question

If the weather forecast was a martingale,

what would be the best prediction for tomorrow?

Merci pour votre attention !