Young researchers between geometry and stochastic analysis

Bergen, Norway

Covariant Riesz transforms and the Calderón-Zygmund inequality

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Introduction

Setting & Notation

Let (M, g) be (geodesically) complete (non-compact) Riemannian manifold of dimension $\dim M = m$ (w/o boundary) and denote by vol_g its volume measure.

- $\mathbf{d} : C^\infty(M) \to \boldsymbol{\Gamma}_{C^\infty}(T^*M)$ exterior derivative on 0-forms, i.e. functions
- $\mathbf{d}^{(1)} : \boldsymbol{\Gamma}_{C^\infty}(T^*M) \to \boldsymbol{\Gamma}_{C^\infty}(\bigwedge^2 T^*M)$ exterior derivative on 1-forms
- $\Delta = \mathbf{d}^*\mathbf{d}$ the Laplace-Beltrami operator on $L^2(M)$! ess. s.a.
- $\Delta^{(1)} = (\mathbf{d}^{(1)})^*\mathbf{d}^{(1)} + \mathbf{d}\mathbf{d}^*$ Hodge-Laplace operator on $\boldsymbol{\Gamma}_{L^2}(T^*M)$! ess. s.a.
- $\nabla : \boldsymbol{\Gamma}_{C^\infty}(T^*M) \to \boldsymbol{\Gamma}_{C^\infty}(T^*M \otimes T^*M)$ covariant derivative
- X a $\mathrm{BM}(M, g, x)$ with maximal lifetime $\zeta(x)$ with
- (stochastic) parallel transport $/\!/_t$

Motivation

Example

By the spectral theorem and the essential self-adjointness of $\Delta^{(1)}$,

$$\begin{aligned}\forall f \in C^\infty(M): \quad \left\|\operatorname{Hess} f\right\|_p &\overset{\text{def}}{=} \left\|\nabla \mathbf{d}^{(1)} f\right\|_p \\ &= \left\|\nabla(\Delta^{(1)} + \lambda)^{-1/2}\mathbf{d}^{(1)}(\Delta + \lambda)^{-1/2}(\Delta^{(1)} + \lambda) f\right\|_p \\ &\leqslant \left\|\nabla(\Delta^{(1)} + \lambda)^{-1/2}\right\|_p \left\|\mathbf{d}^{(1)}(\Delta + \lambda)^{-1/2}\right\|_p \left(\left\|\Delta f\right\|_p + \lambda \left\|f\right\|_p\right)\end{aligned}$$

$$\exists ? C > 0: \quad \left\|\operatorname{Hess} f\right\|_p \leqslant C\left(\left\|\Delta f\right\|_p + \left\|f\right\|_p\right). \tag{CZ(p)}$$

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$$= \|\nabla(\Delta^{(1)} + \lambda)^{-1/2} \mathbf{d}^{(1)} (\Delta + \lambda)^{-1/2} (\Delta^{(1)} + \lambda) f\|_p$$
$$\leqslant \underbrace{\|\nabla(\Delta^{(1)} + \lambda)^{-1/2}\|_p}_{?} \|\mathbf{d}^{(1)}(\Delta + \lambda)^{-1/2}\|_p \left(\|\Delta f\|_p + \lambda \|f\|_p\right)$$

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Want establish L^p boundedness of the covariant Riesz transform

$$\forall \lambda > 0 : \qquad \left\| \nabla (\Delta^{(1)} + \lambda)^{-1/2} \right\|_p < \infty \tag{∇RT}$$

Bakry [1] (1987) L^p boundedness of «classical» Riesz transform

$$\forall \lambda > 0 : \qquad \left\| \mathbf{d}^{(1)} (\Delta^{(1)} + \lambda)^{-1/2} \right\|_p < \infty \tag{RT}$$

Only needs $\mathrm{Ric} \geqslant -C$ for some $C > 0$

Note Laplace-Beltrami operator commutes with the exterior differential, but not with the Levi-Civita connection ∇.

In fact Easily (∇RT) $\Longrightarrow$ (RT) (as $\mathbf{d}$ is the anti-symmtrisation of ∇)

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How to prove (∇RT)?

Recalling that the Laplace transform is given by

$$\nabla(\Delta^{(1)} + \lambda)^{-1/2} = \int_0^\infty \nabla \mathrm{e}^{-t/2\Delta^{(1)}} t^{-1/2} \mathrm{e}^{-t\lambda} \mathrm{d}t$$

there is a highly sophisticated machinery from harmonic analysis (even on metric measure spaces) by Auscher, Coulhon et al. (2004) showing that (∇RT) follows from

$$\exists C > 0 \; \forall t > 0 : \qquad \left\| \nabla e^{-t\Delta^{(1)}} \right\|_p < C \mathrm{e}^{Ct} t^{-1/2} \tag{SG}$$

an estimate for the heat semigroup.

Idea use probabilistic path integral formulae for $\nabla e^{-t\Delta^{(1)}}$ to prove (SG)
$\hat{=}$ Bismut-type formulae

Probabilistic Approach

Bismut-type formulae

Provide derivative formulae of heat (diffusion) semigroups on manifolds.

Introduced by Bismut [2] 1984, extended to various frameworks: Notably, by Elworthy & Li [5, 6] and the general setting using martingale methods for sections of vector bundles Driver & Thalmaier [4] in 2001.

Also results for jump diffusions [3], [10], subordinated Brownian motion, α-stable processes [12] and Lévy processes [9], [11] on $\mathbb{R}^n$.

Curvature & Parallel Transport

Ricci tensor

$$\mathrm{Ric} \in \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M)$$

$$\forall x \in M: \quad \mathrm{Ric}_x : \mathrm{T}_xM \times \mathrm{T}_xM \to \mathbb{R} \quad \text{bilinear form}$$

and let

$$\mathrm{Ric}_{/\!/_t} := /\!/_t^{-1} \circ \mathrm{Ric}_{X_t} \circ /\!/_t \in \mathrm{End}(\mathrm{T}M),$$

where $/\!/_t : \mathrm{T}_xM \to \mathrm{T}_{X_t}M$ is the parallel transport along our diffusion $X_t = X_t(x)$ started at $x \in M$:

$$\begin{array}{ccc} \mathrm{T}_xM & \overset{\mathrm{Ric}_{/\!/_t}}{\dashrightarrow} & \mathrm{T}_xM \\ {\scriptstyle /\!/_t}\downarrow & & \uparrow{\scriptstyle /\!/_t^{-1}} \\ \mathrm{T}_{X_t}M & \underset{\mathrm{Ric}_{X_t}}{\longrightarrow} & \mathrm{T}_{X_t}M \end{array}$$

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$$\begin{CD}
\mathrm{T}_xM @>{\mathrm{Ric}_{/\!/_t}}>> \mathrm{T}_xM \\
@V{/\!/_t}VV @AA{/\!/_t^{-1}}A \\
\mathrm{T}_{X_t}M @>>{\mathrm{Ric}_{X_t}}> \mathrm{T}_{X_t}M
\end{CD}$$

Curvature & Damped Parallel Transport

For all $x \in M$, we define linear transformation Q_t as solution to the pathwise ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t = -\frac{1}{2} Q_t \operatorname{Ric}_{/\!/_t}$$

$$Q_0 = \mathrm{id}_{\mathsf{T}_x M},$$

$\operatorname{Ric}_{/\!/_t} = /\!/_t^{-1} \circ \operatorname{Ric}_{X_t} \circ /\!/_t$ along the paths of $X(x)$. We consider

$$Q_t \circ /\!/_t^{-1} : \mathsf{T}_{X_t} M \to \mathsf{T}_x M$$

the damped parallel transport along X_t.

Theorem (Probabilistic formulae)

Let f be bounded and $u(x,t) = P_t f(x)$ be the (minimal) solution to the heat equation

$$\partial_t u = \Delta u, \quad u(\cdot, 0) = f.$$

- *(Semigroup formula)* $P_t f(x) = \mathbb{E}^x\left(f(X_t)\mathbb{1}_{\{t<\zeta\}}\right) = \mathbb{E}\left(f(X_t(x))\mathbb{1}_{\{t<\zeta(x)\}}\right)$
- *(Derivative formula) If $f \in C_b^1(M)$ and Ric bounded below,*

$$\nabla P_t f(x) = \mathbb{E}^x\left(Q_t /\!/_t^{-1} \nabla f(X_t)\right)$$

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- *(Bismut formula) If* $f \in \mathscr{B}_b(M)$ *and no assumption on* Ric,

$$\forall v \in T_x M: \quad \langle \nabla_x P_t f, v \rangle = -\mathbb{E}^x \left(f(X_t) \mathbb{1}_{\{t<\zeta\}} \int_0^\tau \left\langle Q_s \dot{\ell}_s, dB_s \right\rangle \right)$$

where

- Q *linear transform from last slide depending on the Ricci curvature*
- $\tau = \tau_D(x) \wedge t$ *the first exit time of* $X_t(x)$ *from some relatively compact neighbourhood* D *of* x
- B *is a Brownian motion in* $T_x M$*, the antidevelopment of* X *by* $dB = /\!\!/^{-1} \circ dX$
- ℓ *is any adapted process in* $T_x M$ *with absolutely continuous paths of finite energy such that* $\ell_0 = v$, $\ell_\tau = 0$

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Covariant Feynman-Kac formula

As a linear operator on the space of all L^2-Borel sections on (M, g),

$$(P_t)_{t>0} := \left(e^{-t/2\Delta^{(1)}}\right)_{t>0} \subset \mathscr{L}(\Omega^1_{L^2}(M))$$

is the heat semigroup defined by spectral calculus. For all $t > 0$ and every $x \in M$,

$$\forall \alpha \in \Omega^1_{L^2}(M): \quad e^{-t/2\Delta^{(1)}}\alpha(x) \equiv P_t\alpha(x) = \int_M p_t(x, y)\alpha(y)\,\mathrm{vol}_g(\mathrm{d}y).$$

Malliavin 1978, Driver/Thalmaier 2001 [4], Güneysu [7] 2012

$$\forall \alpha \in \Gamma_{C_c^\infty}(T^*M),\ t \geqslant 0: \quad e^{-t/2\Delta^{(1)}}\alpha(x) = \mathbb{E}^x\left(Q_t /\!\!/_t^{-1}\alpha(X_t)\right)$$

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Strategy

find local martingale $\xrightarrow{\text{bounded}}$ (true) martingale

$$\xrightarrow[\text{at endpts}]{\text{taking } \mathbb{E}} \quad \mathbb{E}N_t = \mathbb{E}N_0 \equiv e^{-t/2\Delta^{(1)}}\alpha(x).$$

1^o **By Itô's formula** for fixed $t > 0$

$$N := Q /\!\!/^{-1} e^{(t-\bullet)/2\Delta}\alpha(X(x)) \quad \text{is a local martingale.}$$

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3^o Taking expectation at the endpoints

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$(\ell_s)_{s\in[0,t]}$ a bounded adapted process with paths in the Cameron-Martin space $\mathrm{L}^{1,2}([0,t], \mathrm{T}_xM \otimes \mathrm{T}_xM)$ and

$$\int_0^s |\dot{\ell}_r|^2 \,\mathrm{d}r < \infty \qquad \text{such that} \qquad \ell_0 = \xi \quad \text{and} \quad \ell_s = 0 \qquad s \geqslant t \wedge \tau.$$

Typically $\quad \ell_s := \frac{t-s}{t}\xi$

We define two sections

$$\underline{\mathrm{Ric}} \in \Gamma_{C^\infty}(\mathrm{End}\,\mathrm{T}^*M^{\otimes 2}) \qquad \text{and} \qquad \rho \in \Gamma_{C^\infty}(\mathrm{Hom}(\mathrm{T}^*M, \mathrm{T}^*M^{\otimes 2}))$$

as a «suitable» difference of Ric, and ∇ Ric, , respectively.

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as a «suitable» difference of Ric, Riem and ∇ Ric, ∇ Riem, respectively.

And we set

$$\ell_s^{(1)} := \int_0^s (Q_r)^{-1}(\mathrm{d}B_r \lrcorner \tilde{Q}_r\dot{\ell}_r), \qquad \ell_s^{(2)} := \frac{1}{2}\int_0^s (Q_r)^{-1}\rho_{/\!/_r}\tilde{Q}_r\ell_r\,\mathrm{d}r, \qquad s \geqslant t \wedge \tau.$$

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Theorem (B/Güneysu, 2018)

Let $|\mathrm{Riem}|, |\nabla\, \mathrm{Riem}| \leqslant A$ *for some* $A < \infty$*. Then*

$$\forall \alpha \in \Gamma_{C^\infty}(\mathrm{T}^* M),\ t > 0,\ x \in M,\ \xi \in \mathrm{T}_x M^{\otimes 2}:$$

$$\left\langle \nabla e^{-\Delta^{(1)} t/2} \alpha(x), \xi \right\rangle = -\mathbb{E}^x \left\langle Q_t /\!\!/_t^{-1} \alpha(X_t), \ell_t^{(1)}(\xi) + \ell_t^{(2)}(\xi) \right\rangle$$

Proof.

Essentially by Itô's formula, the process

$$N_s := \left\langle \tilde{Q}_s /\!\!/_s^{-1} \nabla e^{-(t-s)/2\Delta^{(1)}} \alpha(X_s), \ell_s(\xi) \right\rangle - \left\langle Q_s /\!\!/_s^{-1} e^{-(t-s)/2\Delta^{(1)}} \alpha(X_s), \ell_s^{(1)}(\xi) + \ell_s^{(2)}(\xi) \right\rangle$$

is a local martingale. Showing N is a true martingale, the claim follows by taking expectations. □

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find local martingale $\xRightarrow{\text{bounded}}$ (true) martingale

$\xRightarrow[\text{at endpts}]{\text{taking } \mathbb{E}}$ $\mathbb{E} N_t = \mathbb{E} N_0 \equiv e^{-t/2\Delta^{(1)}} \alpha(x).$

Lemma (boundedness of N)

Let $|\mathrm{Riem}|, |\nabla\,\mathrm{Riem}| \leqslant A$ *for some* $A < \infty$. *Let* $1 \leqslant q < \infty$, $t > 0$ *and* $\xi \in \mathrm{T}_x M$ *(*$x \in M$*). Then:*

(a) *We have*

$$\left(\mathbb{E}\sup_{s\leqslant t}\left|\ell_s^{(1)}(\xi)\right|^q\right)^{1/q} \leqslant C(q,m)t^{-1/2}e^{tC(A,q,m)}\left|\xi\right|,$$

(b) *and*

$$\left(\mathbb{E}\sup_{s\leqslant t}\left|\ell_s^{(2)}(\xi)\right|^q\right)^{1/q} \leqslant Ce^{C(A,m)t}\left|\xi\right|.$$

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t = -\frac{1}{2} Q_t \operatorname{Ric}_{/\!/_t}, \quad \ell_t^{(1)} = \int_0^t (Q_s)^{-1} (\mathrm{d}B_s \lrcorner \tilde{Q}_s \dot{\ell}_s) \qquad \ell_t^{(2)} = \frac{1}{2} \int_0^t (Q_s)^{-1} \rho_{/\!/_s} \tilde{Q}_s \ell_s \, \mathrm{d}s$$

Proof.

By Gronwall's inequality,

$$|Q_s|, |Q_s^{-1}|, |\tilde{Q}_s|, |\tilde{Q}_s^{-1}| \leqslant \mathrm{e}^{C(A,m)s} \qquad \mathbb{P}\text{-a.s. on } \{s \leqslant t\}.$$

So obviously,

$$\left(\mathbb{E} \sup_{s \leqslant t} \left| \ell_s^{(2)}(\xi) \right|^q \right)^{1/q} \leqslant C e^{tC(A,m)q} \left| \xi \right|^q .$$

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By the Burkholder-Davis-Gundy inequality and a well-known procedure to estimate the Cameron-Martin space valued process, we estimate $\ell^{(1)}$

$$\left(\mathbb{E} \sup_{s \leqslant t} \left|\ell_s^{(1)}(\xi)\right|^q \right)^{1/q} \leqslant C(q,m) \mathbb{E} \left(\int_0^t \left|Q_s^{-1}\right|^2 \left|\tilde{Q}_s\right|^2 \left|\dot{\ell}_s\right|^2 \, \mathrm{d}s \right)^{q/2}$$

$$\leqslant C(q,m) t^{-q/2} e^{tC(A,q,m)} \left|\xi\right|^q . \qquad \square$$

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Theorem (B/Güneysu, 2018)

Let $|\mathrm{Riem}|, |\nabla\,\mathrm{Riem}| \leqslant A$ *for some* $A < \infty$. *Then, for all* $1 < p < \infty$, *there is a constant* $C = C(A, p, m)$ *such that*

$$\forall t > 0: \qquad \left\| \nabla e^{-t/2\Delta^{(1)}} \right\|_p \leqslant C e^{Ct} t^{-1/2}.$$

Proof.

Using the previous Theorem,

$$\left|\nabla e^{-t/2\Delta^{(1)}} \alpha(x)\right|^p \leqslant C(A,p,m) e^{tC(A,p,m)} t^{-p/2} \mathbb{E}^x \left|\alpha(X_t)\right|^p$$

$$\Longrightarrow \int \left|\nabla e^{-t/2\Delta^{(1)}} \alpha(x)\right|^p \mathrm{vol}(dx) \leqslant C(A,p,m) e^{tC(A,p,m)} t^{-p/2} \int \mathbb{E}^x \left|\alpha(X_t)\right|^p \mathrm{vol}(dx)$$

$$\leqslant C(A,p,m) e^{tC(A,p,m)} t^{-p/2} \int\int e^{-t/2\Delta^{(1)}}(x,y)\, \mathrm{vol}(dx)\, |\alpha|^p (y)\, \mathrm{vol}(dy),$$

as $e^{-t/2\Delta^{(1)}}$ is a contraction in $L^r(M)$ for all $r \in [1, \infty]$. Hence,

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$$\leqslant C(A,p,m) e^{tC(A,p,m)} t^{-p/2} \underbrace{\int\int e^{-t/2\Delta^{(1)}}(x,y)\,\mathrm{vol}(\mathrm{d}x)}_{\leqslant 1} |\alpha|^p (y)\,\mathrm{vol}(\mathrm{d}y),$$

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Tusen takk for oppmerksomheten!