PhD Seminar - Summer term 2019

An Introduction to Stochastic Differential Geometry

Robert Baumgarth (Université du Luxembourg)

7th March 2019

Stochastic Differential Geometry

Aim Brief overview & introduction

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Key ingredients

- Stochastic processes: Brownian motion & martingales
- Differential Geometry: Smooth (Riemannian) manifolds, flows to a vector field
- Functional Analysis: C_0 contraction semigroups & generators (Hille-Yosida)
- Curvature

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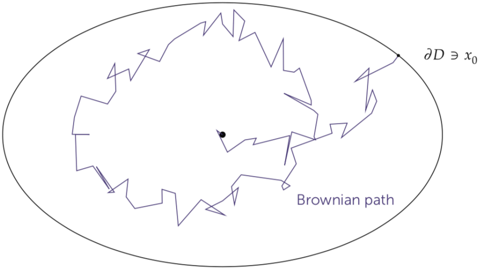
Brownian motion & its generator

Brownian Motion



play

d-dimensional **Brownian motion** is a(n almost surely) continuous stochastic process with stationary and independent increments such that $B_t - B_s \sim \mathcal{N}(0,(t-s))^{\otimes d}$ are normally distributed, i.e., have a Gaussian density

$$p_{t-s}(x,\mathrm{d}y) = \frac{1}{\sqrt{2\pi(t-s)}^d} \exp\left(-\frac{1}{2}\frac{|x-y|^2}{t-s}\right)\mathrm{d}y.$$

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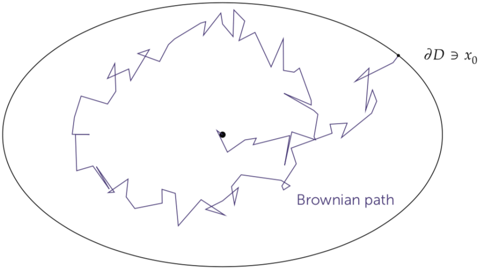

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Infinitesimal Generator

$(\mathbf{A}, \mathscr{D}(\mathbf{A}))$ of a suitable semigroup $(P_t)_{t\geqslant 0}$ on a Banach space is

$$\mathbf{A}u := \lim_{t\to 0} \frac{P_t u - u}{t}. \tag{1}$$

Hence, by Taylor's formula

$$P_t u = u + t\mathbf{A}u + o(t),$$

so that the generator $\mathbf{A}$ gives the first-order approximation to P_t for small t.

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$$P_t u(x) := \mathbb{E}^x u(B_t) = \frac{1}{\sqrt{2\pi t}^d} \int_{\mathbb{R}^d} u(y) \exp\left(-\frac{|x-y|^2}{2t}\right) \mathrm{d}y = u * p_t(x)$$

is a convolution of the heat kernel, thus a solution to the heat equation and

$$\mathbb{E}^x u(B_t) \approx u(x) + \mathbf{A} P_t u(x). \tag{2}$$

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Example Brownian Motion

For simplicity $d = 1$. By Taylor's formula, we get

$$\mathbb{E}^x u(B_t) \approx \mathbb{E}^x\Big[u(x) + u'(x)(B_t - x) + \frac{1}{2}u''(x)(B_t - x)^2\Big] = u(x) + 0 + \frac{1}{2}tu''(x).$$

So, for $d \geqslant 1$, $(\mathbf{A}, \mathscr{D}(\mathbf{A})) = \left(\frac{1}{2}\Delta, \mathscr{C}_\infty\right)$, the Laplacian is the generator of Brownian motion.

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For Brownian motion in higher dimensions, it can still be shown that $\mathbf{A}u = \frac{1}{2}\Delta u$ where $u \in \mathscr{D}\left(\frac{1}{2}\Delta\right)$. However, only $C_\infty \subsetneq \mathscr{D}\left(\frac{1}{2}\Delta\right)$.

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Fundamental Relation

One can show

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t u(x) = \mathbf{A} P_t u(x) = P_t \mathbf{A} u(x),$$

i.e. the generator is the time derivative of the mapping $t \mapsto P_t u(x)$. Hence,

$$\mathbf{A} P_t u(x) - u(x) = \int_0^t P_r \mathbf{A} u(x) \mathrm{d}r$$

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Manifolds

Smooth Riemannian manifolds

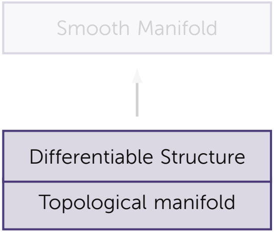
Differentiable Structure

Topological manifold

topological space,

locally Euclidean space

Smooth Riemannian manifolds



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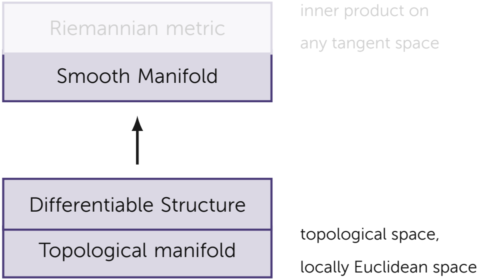

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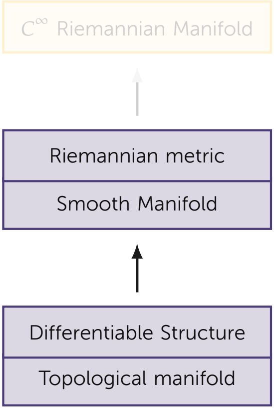

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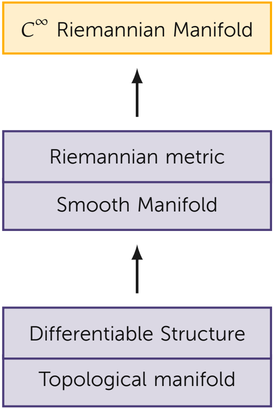

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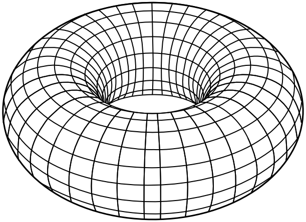
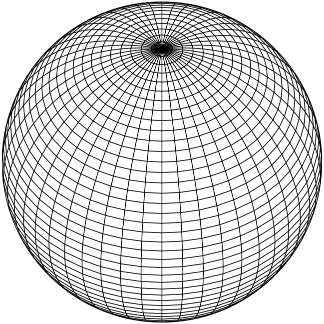
Examples of manifolds





Flow to a vector field

Given vector field A on M, consider the smooth curve $t \mapsto x(t)$ in M:

$$\forall x \in M : \quad \dot{x}(t) = A(x(t))$$
$$x(0) = x_0$$

Then Corresponding flow curve to A at $x : t \mapsto \varphi_t(x) := x(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = A\varphi_t$$
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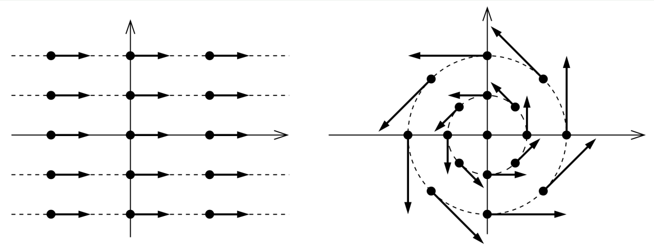
Example

Let (x, y) standard coordinates on $\mathbb{R}^2$.



$A = \partial_x$ $\qquad\qquad$ $A = x\partial_y - y\partial_x$

Flow to a 2nd order differential operator

Let **A** be a second order partial differential operator (PDO) on M, e.g.

$$\mathbf{A} = \sum_{i=1}^{r} A_i^2,$$

where $A_1, \dots, A_r \in \Gamma(TM)$ ($r \in \mathbb{N}$) and $A_i^2 f := A_i(A_i f)$.

Example

Δ, the Laplace operator on $\mathbb{R}^n$, defined as

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Stochastic Flows & the PDE connection

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What are **A**-diffusions good for?

Dirichlet problem (DP)

Let $\emptyset \neq D \subset M$ open, connected, relatively compact domain, $\varphi \in C(\partial D)$ and Δ_M Laplace(-Beltrami) operator on M. Find a function $u \in C(\overline{D}) \cap C^2(D)$

$$\frac{1}{2}\Delta_M u = 0 \text{ in } D$$

$$u = \varphi \text{ on } \partial D.$$

Definition

We define the **first exit time** of D to be

$$\tau(x) := \inf\left\{t \geqslant 0 : B_t(x) \notin D\right\},$$

Now Assume u is solution to (DP).

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SDEs & BM on Manifolds

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$$\left.\begin{array}{c}\text{Principal}\\ \text{Vector}\end{array}\right\}\ \text{bundle} \ \hat{=}\ \text{fibre bundle} \left\{\begin{array}{l} F = (G, \lhd)\ G \text{ Lie group} \\ F = \mathbb{R}\text{-vector space, } \rhd \text{ linear}\end{array}\right.$$

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The pair (A, B) is called **stochastic differential equation on M** (SDE on M), if:

(i) B Brownian motion with values in a finite-dimensional $\mathbb{R}$-vector space E,

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with initial condition $X_0 = x_0$ is a continuous adapted process $(X_t)_{t \geqslant 0}$ with values in M such that for all $f \in C_c^\infty(M)$ the process $f(X)$ is a real semimartingale and satisfies

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Idea

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$$\mathbf{A} = \sum_{i=1}^{r} A_i^2 .$$

Solutions to the SDE above will be stochastic flows to **A**.

But $\nexists$ Hörmander-type representation of Δ_M, i.e. $\nexists$ canonical way of writing Δ_M as sums of squares, if manifold is not parallelisable, i.e. tangent bundle $TM \xrightarrow{\pi} M$ not trivial.

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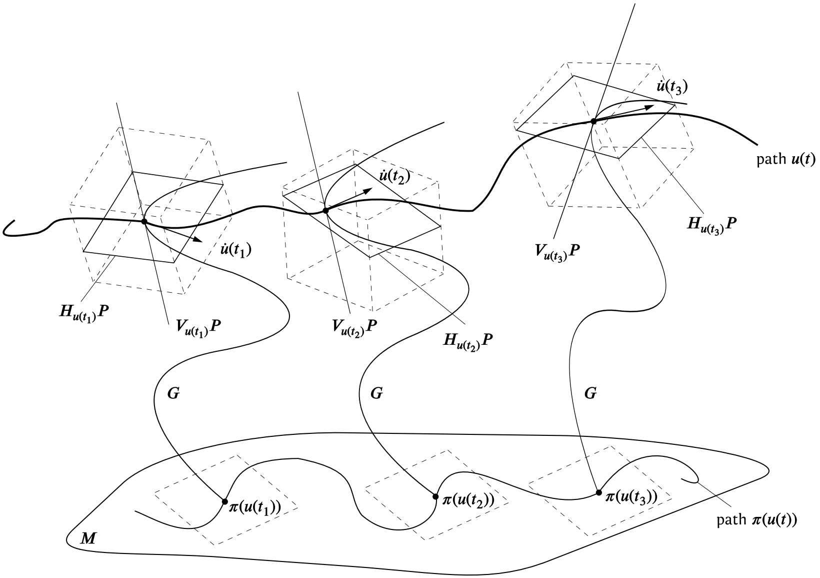

Figure: Horizontal lift $u(t)$ through principal G-bundle

Eells-Elworthy-Malliavin approach

Orthonormal frame bundle: $\mathscr{O}(M) \xrightarrow{\pi} M$ over M

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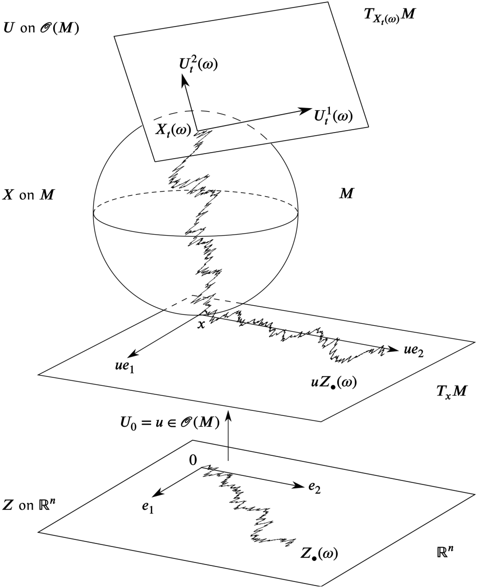
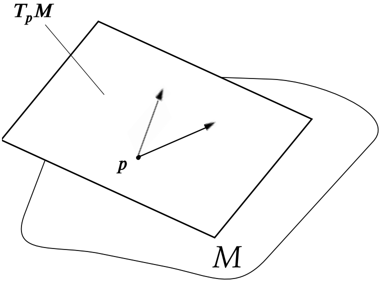
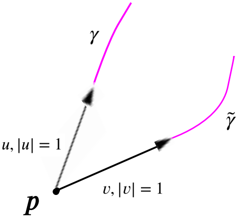
Brownian motion on manifolds & Stochastic development



Curvature

Sectional Curvature





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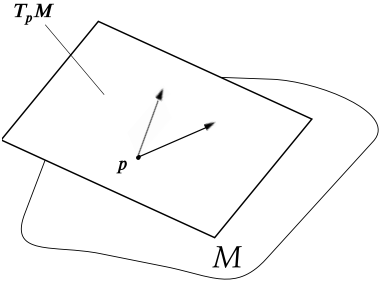

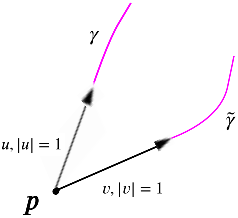

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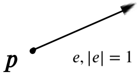

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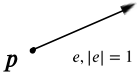

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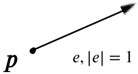

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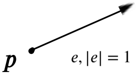

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Outlook

Theorem (Probabilistic formulae)

Let f be bounded and $u(x,t) = P_t f(x)$ be the (minimal) solution to the heat equation

$$\partial_t u = \mathbf{A} u, \quad u(\cdot, 0) = f.$$

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where

- Q *linear transform depending on the Ricci curvature*
- $\tau = \tau_D(x) \wedge t$ *the first exit time of* $X_t(x)$ *from some relatively compact neighbourhood* D *of* x
- B *is a Brownian motion in* $T_x M$*, related to* X *by* $dB = /\!\!/^{-1} \circ dX$*,*
- ℓ *is any adapted process in* $T_x M$ *with absolutely continuous paths of finite energy such that* $\ell_0 = v$, $\ell_\tau = 0$.

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where

- Q *linear transform depending on the Ricci curvature*
- $\tau = \tau_D(x) \wedge t$ *the first exit time of* $X_t(x)$ *from some relatively compact neighbourhood* D *of* x
- B *is a Brownian motion in* $T_x M$*, related to* X *by* $\mathrm{d}B = /\!\!/^{-1} \circ \mathrm{d}X$*,*
- ℓ *is any adapted process in* $T_x M$ *with absolutely continuous paths of finite energy such that* $\ell_0 = v$, $\ell_\tau = 0$.

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Questions?