Infinitesimal Generator of a C_0-contraction semigroups - a probabilistic approach

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Abstract

The aim of this article is to give a brief introduction to the generator of a strongly continuous or C_0-contraction operator-semigroup. Therefore we first recall the necessary definitions from functional analysis. Then we will introduce the infinitesimal generator and give some examples why generators are of interest. The second part describes the powerful meaning of the generator from a probabilistic point of view. Finally, we investigate the famous Brownian motion process. No proofs are given throughout the text. They can be found in [SP14], [Øks07] and [IW81].

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1 Semigroups

First, let us recall some general calculus definitions.

1.1 Definition. A(n operator) **semigroup** $(P_t)_{t\geqslant 0}$ on a Banach space $(L, \|\cdot\|)$ is a family of linear operators $P_t : L \to L$ satisfying:

$$\forall s, t \in L : P_{t+s} = P_t P_s \quad \text{and} \quad P_0 = \mathrm{id} \tag{1.1}$$

We call $(P_t)_{t\geqslant 0}$ **(of class)** C_0 or a **strongly continuous semigroup** if for all $u \in L$

$$\lim_{t\to 0} \left\| P_t u - u \right\| = 0.$$

1.2 Definition. A semigroup $(P_t)_{t\geqslant 0}$ is a **contraction** on $(L, \|\cdot\|)$ if $\left\| P_t u \right\| \leqslant \|u\|$ for all $u \in L$.

1.3 Definition. The **(infinitesimal) generator** of a C_0-contraction semigroup $(P_t)_{t\geqslant 0}$ on $(L, \|\cdot\|)$ is the operator

$$\mathbf{A}u := \lim_{t\to 0} \frac{P_t u - u}{t}, \tag{1.2}$$

where the domain $\mathscr{D}(\mathbf{A})$ of $\mathbf{A}$ is the set of f where the limit in (1.2) exists.

Note that A is densely defined and closed in L. Moreover the generator uniquely determines the semigroup.

By Taylor's formula (1.2) can be alternatively written as

$$P_t u = u + t\mathsf{A}u + o(t),$$

so that the generator A gives the first-order approximation to P_t for small t.

One can show that

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t u(x) = \mathsf{A} P_t u(x) = P_t \mathsf{A} u(x), \tag{1.3}$$

i.e. the generator is the time derivative of the mapping $t \mapsto P_t u(x)$. Reading (1.3) as a partial differential equation we see that $u(t,x) := P_t f(x)$ is a solution of

$$\partial_t u(t,x) = \mathsf{A}u(t,x), \quad u(0,x) = f(x). \tag{1.4}$$

Another reason why the generator is of interest is the famous *Hille-Yosida* theorem.

1.4 Theorem (Hille-Yosida). *A linear operator $(\mathsf{A}, \mathscr{D}(\mathsf{A}))$ on $(L, \|\cdot\|)$ generates a C_0-contraction semigroup if and only if the following conditions hold*

1. *$\mathscr{D}(\mathsf{A})$ is dense in L.*
2. *A is dissipative, i.e. for all $\lambda > 0, u \in \mathscr{D}(\mathsf{A}) : \|\lambda u - \mathsf{A}u\| \geqslant \lambda \|u\|$.*
3. *$\mathscr{R}(\lambda - \mathsf{A}) = L$ for one (then for all) $\lambda > 0$, i.e. $\lambda - \mathsf{A}$ is surjective.*

A simple consequence of this theorem is that the generator A is **maximally dissipative**, i.e. there is no (proper) extension of the operator which is dissipative.

2 The infinitesimal generator in stochastic

From a probabilistic point of view one defines for a Feller process $(X_t)_{t\geqslant 0}$ the transition semigroup $P_t u(x) := \mathbb{E}^x u(X_t)$. So (1.3) becomes

$$\mathsf{A}u(x) = \lim_{t\to 0} \frac{\mathbb{E}^x u(X_t) - u(x)}{t} \tag{2.1}$$

or in other words

$$\mathbb{E}^x u(X_t) \approx u(x) + \mathsf{A}P_t u(x). \tag{2.2}$$

So, essentially, the generator describes the movement of the process in an infinitesimal time interval.

One can show that for every Feller semigroup the limit exists in $(C_\infty, \|\cdot\|_\infty)$ and hence for every Markov process. The more important probabilistic reason is that for every adapted Feller process $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ on $\mathbb{R}^d$ the process

$$M_t^u := u(X_t) - u(x) - \int_0^t \mathsf{A}u(X_r)\mathrm{d}r \quad (u \in \mathscr{D}(\mathsf{A}), t \geqslant 0) \tag{2.3}$$

is an $\mathscr{F}_t$-martingale. Since martingales have constant expectation it follows that

$$\begin{aligned}\mathbb{E}^x u(X_t) - u(x) &= \mathbb{E}^x \int_0^t \mathsf{A}u(X_r)\mathrm{d}r \\ &= \int_0^t \mathbb{E}^x(\mathsf{A}u)(X_r)\mathrm{d}r \\ \iff P_t u(x) - u(x) &= \int_0^t P_r \mathsf{A}u(x)\mathrm{d}r,\end{aligned}$$

from which we easily attain *Dynkin's formula* by Doob's optional stopping theorem: Let σ be an $\mathscr{F}_t$ stopping time with $\mathbb{E}^x\sigma < \infty$ then

$$\mathbb{E}^x u(X_\sigma) - u(x) = \mathbb{E}^x \int_0^\sigma \mathsf{A}u(X_r)\mathrm{d}r \quad (u \in \mathscr{D}(\mathsf{A})). \tag{2.4}$$

This connection allows us to solve classical partial differential equations, like the Dirichlet problem, with martingale methods in an elegant probabilistic way. Finally, we discuss an important example.

2.1 Example. Let $(B_t)_{t\geqslant 0}$ a one-dimensional Brownian motion. First let $d = 1$ and recall that $\mathbb{E}^x(B_t - x) = 0$ and $\mathbb{E}^x(B_t - x)^2 = t$. Then by Taylor's formula

$$\mathbb{E}^x u(B_t) \approx \mathbb{E}^x \left(u(x) + u'(x)(B_t - x) + \frac{1}{2}u''(x)(B_t - x)^2 \right) = u(x) + 0 + \frac{1}{2}tu''(x).$$

So we might assume that it holds $(\mathsf{A}, \mathscr{D}(\mathsf{A})) = \left(\frac{1}{2}\Delta, \mathscr{C}_\infty\right)$.[1] For $d = 1$ that is true. For Brownian motion in higher dimensions, it can still be shown that $\mathsf{A}u = \frac{1}{2}\Delta u$ where $u \in \mathscr{D}\left(\frac{1}{2}\Delta\right)$. However, we only get $C_\infty \subsetneq \mathscr{D}\left(\frac{1}{2}\Delta\right)$.

Referring to (2.3) we see that

$$M_t^u := u(X_t) - u(x) - \int_0^t u''(B_r)\mathrm{d}r \quad (u \in \mathscr{D}(\mathsf{A}), t \geqslant 0)$$

is a martingale. Having Itô's formula in mind, this is not surprising since

$$u(B_t) - u(B_0) = \int_0^t f'(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s = M_t^u + \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s.$$

From (1.3) we see that $u(t, x) := \mathbb{E}^x f(B_t)$ is the unique solution of the heat equation

$$\partial_t u(t, x) = \frac{1}{2}\partial_x^2 u(t, x), \quad u(0, x) = f(x).$$

Let D be an open, bounded and connected domain, $\tau_{D^c} := \inf\{t > 0 : B_t \notin D\}$ the first exit time of the set D and $f : \partial D \to \mathbb{R}$ continuous. If ∂D suffices some regularity condition, then the Dirichlet problem

$$\begin{aligned} \Delta u(x) &= 0 && \forall x \in D \\ u(x) &= f(x) && \forall x \in \partial D \\ u(x) &\text{ continuous} && \forall x \in \overline{D} \end{aligned}$$

has the unique solution $u(t, x) := \mathbb{E}^x f\left(B_{\tau_{D^c}}\right)$.

[1] Where $\mathscr{C}_\infty =: \mathscr{C}_\infty(\mathbb{R}^d)$ denotes the family of all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ vanishing at infinity, i.e. $\lim_{|x|\to\infty} u(x) = 0$, equipped with the uniform norm $\|\cdot\|_\infty$.

References

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