On the Musical Isomorphisms

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May 15, 2015

Abstract

The aim of this article is to give a brief introduction to the so called *musical isomorphisms* and their impact in extending the definitions of vector calculus operators (∇, div, curl) on manifolds. Therefore, we first recall the necessary definitions from tensor calculus and differential geometry. Next, we introduce a method to change the index of a tensor. The second part introduces the musical isomorphisms and the most important sharp-operation, the definition of the gradient in an coordinate invariant way. The third part describes the powerful meaning of differential forms in differential geometry und thus, finally, we are able to generalize the operators in a handsome way. A deep step-by-step introduction to this topic can be found in [Lee12] and brief, but handsome, introduction to calculus on manifolds, especially the meaning of differential forms, can be found in [Spi65].

Keywords: Tensor calculus, differential form, vector calculus, grad, ∇, div, curl, musical isomorphism

2010 Mathematics Subject Classification: Primary: 58A10. Secondary: 46A32, 55R25, 58C05, 58C35.

1 Elements of Tensor Algebra

1.1 Tensors

Tensors provide a unified language to talk about multilinear maps. Thus, they generalize the concepts of vectors and covectors, i.e. elements of the dual space of a given vector space. Simplest examples for tensors are covectors, inner products and determinants. Since manifold theory tries to interpret linear approximations of calculus in a coordinate independent way, tensors are a key ingredient to establish the theory of differential forms. Finally, there are important applications in physics, e.g. in general relativity.

Without further comment, let V be real-valued finite-dimensional vector space with basis $e_1, ..., e_n$ and denote by V^* the corresponding dual space, i.e. the vector space of all linear functionals on V, with basis $\epsilon^1, ..., \epsilon^n$. Moreover, let M be a smooth manifold with or without boundary. We will use the customary abbreviation $\partial_i := \partial_{x^i} := \frac{\partial}{\partial x^1}$, (∂_i) for the basis vectors of the tangent space T_pM and $\left(\mathrm{d}x^i\right)$ for the corresponding dual basis.[1]

Recall that a **tensor of type** (m,n), i.e. **covariant** m**-tensor on** V and **contravariant** n**-tensor on** V, is a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{n \text{ copies}} \times \underbrace{V \times \cdots \times V}_{m \text{ copies}} \to \mathbb{R}.$$

Let us denote by $T^{(m,n)}(V)$ the space of all tensors of type (m,n). Then we have the following identities and shorthands.

$$\begin{aligned}
&T^{(0,0)}(V) = T^0(V^*) = T^0(V) := \mathbb{R},\\
&T^{(0,1)}(V) :=: T^1(V^*) = V^*,\\
&T^{(1,0)}(V) :=: T^1(V) = V,\\
&T^{(0,k)}(V) =: T^k(V^*), \quad T^{(k,0)}(V) =: T^k(V).
\end{aligned}$$

In particular, a 0-tensor is a real number and a covariant 1-tensors is nothing else then a covector.

To extend the notion of vector fields to tensors, one defines **tensor fields** as a section of the so called tensor bundle. The **bundle of mixed tensors of type** (m,n) is defined by

$$T^{(m,n)}TM := \bigsqcup_{p\in M} T^{(m,n)}(T_pM).$$

There are some natural identifications.

$$\begin{aligned}
&T^{(0,0)}TM = T^0T^*M = T^0TM = M \times \mathbb{R},\\
&T^{(0,1)}TM = T^1T^*M = T^*M,\\
&T^{(1,0)}TM = T^1TM = TM,\\
&T^{(0,k)}TM = T^kT^*M, \quad T^{(k,0)}TM = T^kTM.
\end{aligned}$$

This means in particular, any contravariant 1-tensor field is a vector field and any covariant 1-tensor field is covector field. Since a 0-tensor is a real number, a 0-tensor field can be interpreted as a continuous real-valued function.

1.2 Raising and lowering indices

Let (M,g) a Riemannian manifold with Riemannian metric g. Given a tensor of type (m,n) we sometimes want to change the indices. Therefore, we can raise the index to an $(m+1,n-1)$ tensor or lower the index to an $(m-1,n+1)$ tensor. First, an

[1]Note that the notation is justified, since the coordinate covector field is none other than the differential $\mathrm{d}x^i$. For any $\left(x^i\right)$ smooth coordinate chart on an open subset $U \subset \mathbb{R}^n$ and the corresponding coframe $\left(\lambda^i\right)$ on U, we get by the definition of the differential

$$\mathrm{d}x^j\big|_p = \frac{\partial x^j}{\partial x^i}\Big|_p \lambda^i\big|_p = \delta^j_i \lambda^i\big|_p = \lambda^j\big|_p.$$

1.1 Example. Let $X = \left(ct, -x^1, -x^2, -x^3\right)$, i.e. in components X_0, $X_j = -x^j$, and choose a tensor with signature $(+---)$ defined as

$$\alpha_{jk} = \alpha^{jk} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

in components $\alpha_{00} = 1$, $\alpha_{j0} = \alpha_{0j} = 0$, $\alpha_{jk} = -\delta_{jk}$. To raise the index one has to multiply by the tensor, $\alpha^{jk} X_j$, i.e.

$$\begin{aligned} \alpha^{00} X_0 + \alpha^{0k} X_k &= X_0, \\ \alpha^{j0} X_0 + \alpha^{jk} X_k &= -\delta^{jk} X_k = -X_j, \end{aligned}$$

and, thus, $\alpha^{jk} X_j = (ct, x^1, x^2, x^3)$.

2 Musical Isomorphisms

On a Riemmanian manifold (M, g) the Riemannian metric g provides a natural isomorphism between the tangent and the cotangent bundle. Define the (smooth) bundle homomorphism $\hat{g} : TM \to T^*M$ for all $p \in M, v \in T_pM$ by

$$\hat{g}(v)(w) := g_p(v, w) \quad (w \in T_pM).$$

Moreover, $Y \mapsto \hat{g}(X)(Y)$ and $X \mapsto \hat{g}(X)$ are linear over $C^\infty(M)$ and $\hat{g}$ is a smooth covariant k-tensor field on M. Choosing any smooth coordinates (x^i), the metric can be written as $g = g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$. Thus if $X, Y \in \mathscr{X}(M)$ are smooth vector fields, we have

$$\hat{g}(X)(Y) = g_{ij} X^i Y^j,$$

so the covector field $\hat{g}(X)$ has the coordinate expression

$$\hat{g}(X) = g_{ij} X^i\, \mathrm{d}x^j,$$

where we have used Einstein notation. Thus, $\hat{g}$ is the bundle homomorphism whose matrix with respect to coordinate frames for TM and T^*M is the same as the matrix of g itself.

The components of the covector field $\hat{g}(X)$ are denoted by

$$\hat{g}(X) = X_j\, \mathrm{d}x^j, \quad \text{where } X_j := g_{ij} X^i.$$

Therefore, one says that $\hat{g}(X)$ is obtained form X by **lowering an index** and its customary to use the notation $X^\flat$ («X-flat»), like you may recognize it from music. Since $\left(g_{ij}\right)$ is a isomorphism its inverse matrix $\left(g^{ij}\right)$ is the matrix of the inverse map $(\hat{g})^{-1} : T^*M \to TM$. $\left(g^{ij}\right)$ is also symmetric and

$$g^{ij} g_{jk} = g_{jk} g^{ij} = \delta_{ik}.$$

Thus, for a smooth covector field $\omega \in \mathscr{X}^*(M)$, the vector field $\hat{g}^{-1}(\omega)$ has the coordinate expression

$$\hat{g}^{-1}(\omega) = \omega^i \partial_{x^i}, \quad \text{where } \omega^i := g^{ij}\omega_j.$$

$\omega^\sharp := \hat{g}^{-1}(\omega)$ («ω-sharp») and we say that $\omega^\sharp$ is obtained from ω by **raising an index**. Because the symbols $\flat$ and $\sharp$ are borrowed from musical notation, these two inverse isomorphisms are frequently called the **musical isomorphisms**.

Thus, $\flat : TM \to T^*M$ sends a vector X to the covector $X^\flat$ and $\sharp : T^*M \to TM$ sends a covector ω to a vector.

2.1 Coordinate independent notation of the gradient on manifolds

The most important $\sharp$-operation is the generalization of the gradient as a vector field on manifolds. For all smooth functions f on M define a vector field, **gradient of** f, by

$$\nabla f =: \operatorname{grad} f := (\mathrm{d}f)^\sharp = \hat{g}^{-1}(\mathrm{d}f).$$

Unraveling the definitions

$$\langle h, \nabla f \rangle_g = \hat{g}(\nabla f)(X) = \mathrm{d}f(X) = Xf.$$

Hence, ∇f is the unique vector field that satisfies

$$\langle \nabla f, X \rangle_g = Xf \quad (X \in \mathscr{X}(M)),$$

and this means

$$\langle \nabla f, \cdot \rangle_g = \mathrm{d}f.$$

In smooth coordinates ∇f has the expression

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j},$$

where we note that ∇f is smooth. In particular, on $\mathbb{R}^n$ this reduces to the well-known formula from classical calculus

$$\nabla f = \delta^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

2.1 Example. Let f be a smooth function in $\mathbb{R}^2$ with respect the Euclidean metric $\overline{g} = \mathrm{d}x^2 + \mathrm{d}y^2$. We compute the gradient in polar coordinates.

The Euclidean metric in polar coordinates on $\mathbb{R}^2$ is given by $\overline{g} = \mathrm{d}r^2 + r\ \mathrm{d}\phi^2$.[2] Thus,

$$\hat{g}(r,\phi) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \Longrightarrow \hat{g}^{-1}(r,\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}.$$

Hence, the gradient is given by

$$\nabla f = \frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial f}{\partial \phi}\frac{\partial}{\partial \phi}.$$

3 Differential Forms and Vector Calculus on Manifolds

The theory of covariant alternating k-tensors is the theory of **differential forms**. Thus, they are (essentially) a generalizations of covector fields. Like covector fields, differential forms can be integrated over curves in a coordinate invariant way. Let us recall briefly the necessary definitions. A k-tensor α on V is called **alternating**, if for all pairs of distinct $i, j \in \mathbb{N}$

$$\alpha(v_1, ..., v_i, ..., v_j, ..., v_n) = -\alpha(v_1, ..., v_j, ..., v_i, ..., v_n).$$

The subspace of all alternating covariant k-tensors on V is denoted by $\Lambda^k(V^*) \subset T^k(V^*)$. Define a projection, called **alternation**,

$$\begin{aligned} \mathrm{Alt} : T^k(V^*) &\to \Lambda^k(V^*) \\ \alpha &\mapsto \frac{1}{k!}\sum_{\alpha \in S_k} (\operatorname{sgn}\alpha)\,({}^\sigma\alpha)\,, \end{aligned}$$

where S_k denotes the symmetric group of k elements. Next, we define the **elementary alternating tensor** $\epsilon^I \in \Lambda^k(V^*)$ by

$$\epsilon^I(v_1, ..., v_k) = \begin{pmatrix} \epsilon^{i_1}(v_1) & \dots & \epsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \epsilon^{i_k}(v_1) & \dots & \epsilon^{i_k}(v_k) \end{pmatrix} = \det \begin{pmatrix} v_1^{i_1} & \dots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_k^{i_1} & \dots & v_k^{i_k} \end{pmatrix}$$

Now, let $\dim V = n$ and $\left(\epsilon^I\right)$ be a basis vor V^*. Then, it is easy to see that a basis for $\Lambda^k(V^*)$ is given by

$$\mathscr{E} = \left\{\epsilon^I \,:\, I \in \mathbb{N}^k \text{ an increasing multiindex}\right\},$$

[2] This is easily seen by computing the pullback of $\overline{g}$ w.r.t the Euclidean metric for $F = \mathrm{id}$ and substituting $x = r\cos\phi$, $y = r\sin\phi$:

$$\begin{aligned} \overline{g} &= \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}(r\cos\phi)^2 + \mathrm{d}(r\sin\phi)^2 \\ &= (\cos\phi\ \mathrm{d}r - r\sin\phi\ \mathrm{d}\phi)^2 + (\sin\phi\ \mathrm{d}r + r\cos\phi\ \mathrm{d}\phi)^2 \\ &= \mathrm{d}r^2 + r\ \mathrm{d}\phi^2 \end{aligned}$$

and therefore $\dim \Lambda^k(V^*) = \binom{n}{k}$. In particular, if $k > n$, then $\dim \Lambda^k(V^*) = 0$ and $\dim \Lambda^k(V^*) = 1$.

For any $\omega \in \Lambda^k(V^*)$, $\eta \in \Lambda^l(V^*)$, we define the **wedge product** or **exterior product** to the $(k+l)$-covector[3]

$$\omega \wedge \eta := \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta).$$

The wedge product is bilinear, associative and anticommutative. Moreover, for any multiindex $I = (i_1, ..., i_k) \in \mathbb{N}^k$

$$\epsilon^{i_1} \wedge ... \wedge \epsilon^{i_k} = \epsilon^I,$$

so we will use both elements interchangeable. For any other multiindex $J = (j_1, ..., j_l) \in \mathbb{N}^l$ we have $\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$, where the composition is defined as $IJ = (i_1, ..., i_k, j_1, ..., j_l)$. For any $\omega^1, ..., \omega^k \in V^*$, $v_1, ..., v_k \in V$

$$\omega^1 \wedge ... \wedge \omega^k(v_1, ..., v_k) = \det \omega^j(v_i).$$

3.1 Example. The dual basis for $\left(\mathbb{R}^3\right)^*$ is given by (e^1, e^2, e^3). Since $e^j(e_i) = \delta_i^j$, we have

$$\begin{aligned} e^{13}(v, w) &= \begin{vmatrix} e^1(v) & e^1(w) \\ e^3(v) & e^3(w) \end{vmatrix} = v^1 w^3 - w^1 v^3, \\ e^{123}(v, w, u) &= \det(v, w, u). \end{aligned}$$

Next, we want to relate vectors with alternating tensors. For each $v \in V$, we define a linear map, called **interior multiplication by** v, by

$$\begin{aligned} i_v &: \Lambda^k(V^*) \to \Lambda^{k-1}(V^*) \\ i_v\omega(w_1, ..., w_{k-1}) &:= \omega(v, w_1, ..., w_{k-1}). \end{aligned}$$

By convention, $i_v\omega = 0$, for any 0-covector ω. As $i_v\omega$ is obtained from ω by inserting v into the first slot, one says «v into ω» and denotes $v \lrcorner \omega$. More precisely,

$$v \lrcorner \left(\omega^1 \wedge ... \wedge \omega^k\right) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v) \omega^1 \wedge ... \wedge \overline{\omega^i} \wedge ... \wedge \omega^k, \tag{3.1}$$

where the overlined element $\overline{\omega^i}$ marks that it is omitted.

Like tensor fields, one defines in a natural way a smooth subbundle and vector bundle of rank $\binom{n}{k}$ over M as the subset of $T^k T^* M$ of alternating tensors

$$\Lambda^k T^* M := \bigsqcup_{p \in M} \Lambda^k(T_p^* M).$$

[3]Note that the coefficient is chosen only for computational reasons.

A section of $\Lambda^k T^*M$ is said to be a **(differential)** k**-form**, i.e. a (continuous) tensor field, whose values at each point is an alternating tensor. The vector space of all smooth k-forms is denoted by $\Omega^k(M)$. The wedge product is extended pointwise: $(\omega\wedge\eta)_p := \omega_p\wedge\eta_p$ and, since $\Omega^0(M) = C^\infty(M)$, we use the shorthand $f \wedge \eta =: f\eta$. Thus, for any smooth coordinate chart, ω can locally be written as

$$\omega = \sum_I \omega_I \,\mathrm{d}x^I = \sum_I \omega_I \,\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k} \quad (I \text{ increasing index}).$$

The interior multiplication also extends naturally acting pointwise: For a vector field X and $\omega \in \Omega^k(M)$, define

$$\begin{aligned} X \lrcorner\, \omega &:= i_X\omega \in \Omega^{k-1}(M) \\ (X \lrcorner\, \omega)_p &:= X_p \lrcorner\, \omega_p. \end{aligned}$$

Now, we are able to give a generalization of the differential of functions. On an manifold M with or without boundary, there exists a unique operator $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ for all $k \in \mathbb{N}_0$, the **exterior derivative**, satisfying

(i) d is linear over $\mathbb{R}$.

(ii) If $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$, then

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k\omega \wedge \mathrm{d}\eta.$$

(iii) $\mathrm{d} \circ \mathrm{d} = 0$.

(iv) For any $f \in \Omega^0(M)$ with differential $\mathrm{d}f$ of f, we have

$$\mathrm{d}f(X) = Xf.$$

In any smooth chart d is given by

$$\mathrm{d}\left(\sum_I \omega_I \,\mathrm{d}x^I\right) := \sum_I \mathrm{d}\omega_I \wedge \mathrm{d}x^I \quad (I \text{ increasing index}), \tag{3.2}$$

where $\mathrm{d}\omega_I$ is the differential of the function ω_I.

If we choose $w \in \Omega^1(M)$, then, using the anticommutativity of differential forms,

$$\begin{aligned} \mathrm{d}\left(\omega_j \,\mathrm{d}x^j\right) &= \sum_{i,j} \frac{\partial w_j}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \sum_{i<j} \frac{\partial w_j}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^j + \sum_{i>j} \frac{\partial w_i}{\partial x^j} \,\mathrm{d}x^j \wedge \mathrm{d}x^i \\ &= \sum_{i<j} \left(\frac{\partial w_j}{\partial x^i} - \frac{\partial w_i}{\partial x^j}\right) \mathrm{d}x^i \wedge \mathrm{d}x^j, \end{aligned}$$

which reduces for any function $f \in \Omega^0(M)$ to

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i,$$

i.e. the usual differential of f.

Finally, we are able to extend the well-known differential operators of vector calculus to a coordinate independent definition on an arbitrary manifold M. The interior multiplication yields a map

$$\begin{aligned}\beta : \mathcal{X}(\mathbb{R}^n) &\to \Omega^2(\mathbb{R}^n)\\ \beta(X) &:= X \lrcorner \left(\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n\right),\end{aligned}$$

which is linear over $C^\infty(\mathbb{R}^n)$. Moreover, define a bundle isomorphism

$$\begin{aligned}* : C^\infty(\mathbb{R}^n) &\to \Omega^3(\mathbb{R}^n)\\ *(f) &:= f\,\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n.\end{aligned}$$

Then the following diagram commutes.

$$\begin{tikzcd}
C^\infty(\mathbb{R}^n) \arrow[r, "\nabla"] \arrow[d, "\mathrm{id}"'] & \mathcal{X}(\mathbb{R}^n) \arrow[r, "\mathrm{curl}"] \arrow[d, "\flat"'] & \mathcal{X}(\mathbb{R}^n) \arrow[r, "\mathrm{div}"] \arrow[d, "\beta"'] & C^\infty(\mathbb{R}^n) \arrow[d, "*"'] \\
\Omega^0(\mathbb{R}^n) \arrow[r, "\mathrm{d}"'] & \Omega^1(\mathbb{R}^n) \arrow[r, "\mathrm{d}"'] & \Omega^2(\mathbb{R}^n) \arrow[r, "\mathrm{d}"'] & \Omega^3(\mathbb{R}^n)
\end{tikzcd}$$

In particular, $\nabla \circ \mathrm{curl} = 0$ and $\mathrm{curl} \circ \mathrm{div} = 0$.

Proof. The first and third commutation is easily seen by direct calculation.

$$(\flat \circ \nabla)(f) = \flat \circ \frac{\partial f}{\partial x^i}\partial_{x^i} = \frac{\partial f}{\partial x^i}\mathrm{d}x^i = \mathrm{d}f.$$

Moreover,

$$\begin{aligned}(\mathrm{d} \circ \beta)(X) &= \mathrm{d}(X \lrcorner \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n)\\
&\overset{(3.1)}{=} \mathrm{d}\Big(\sum_{i=1}^n (-1)^{i-1} \underbrace{\mathrm{d}x^i(X)}_{=X^i} \mathrm{d}x^1 \wedge \dots \wedge \overline{\mathrm{d}x^i} \wedge \dots \wedge \mathrm{d}x^n\Big)\\
&= \sum_{i=1}^n (-1)^{i-1} \sum_{j=1}^n \frac{\partial X^i}{\partial x^j}\, \mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \dots \wedge \overline{\mathrm{d}x^i} \wedge \dots \wedge \mathrm{d}x^n\\
&= \sum_{i=1}^n (-1)^{i-1} \frac{\partial X^i}{\partial x^i}\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n\\
&= *\left(\sum_{i=1}^n \frac{\partial X^i}{\partial x^i}\right)\\
&= (* \circ \mathrm{div})(X).\end{aligned}$$

Last, we show that the curl operator for vector fields coincides with our classical definition. Let $X \in \mathscr{X}(\mathbb{R}^3)$, then we have

$$\begin{aligned}
(\beta \circ \mathrm{curl})(X) &= \beta \left[\left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3} \right) \partial_{x^1} + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1} \right) \partial_{x^2} + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2} \right) \partial_{x^3} \right] \\
&= \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\frac{\partial X^3}{\partial x^1} - \frac{\partial X^1}{\partial x^3} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^3 \\
&\quad + \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3} \right) \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&= \mathrm{d}\left(X^1 \, \mathrm{d}x^1 + X^2 \, \mathrm{d}x^2 + X^3 \, \mathrm{d}x^3 \right) \\
&= (\mathrm{d} \circ \flat)(X).
\end{aligned}$$

Since the diagram and $\mathrm{d} \circ \mathrm{d} = 0$, clearly: $\nabla \circ \mathrm{curl} = 0$ and $\mathrm{curl} \circ \mathrm{div} = 0$. ■

A vector field $X \in \mathscr{X}(M)$ has a corresponding 1-form, just its coordinate expression, $X = X^i \partial_i$. Hence, there is another way to define the curl by

$$\mathrm{curl}\, X := \left(\star (\mathrm{d}X^\flat) \right)^\sharp,$$

where $\star$ is the Hodge-operator, e.g. in the case $n = 3$, one finds

$$\begin{aligned}
\star \mathrm{d}x &= \mathrm{d}y \wedge \mathrm{d}z \\
\star \mathrm{d}y &= \mathrm{d}z \wedge \mathrm{d}x \\
\star \mathrm{d}z &= \mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

and therefore,

$$\begin{aligned}
\sharp \circ \star \circ \mathrm{d} \circ \flat(X) &= \sharp \circ \star \circ \mathrm{d}\left(X^1 \, \mathrm{d}x^1 + X^2 \, \mathrm{d}x^2 + X^3 \, \mathrm{d}x^3 \right) \\
&= \sharp \circ \star \Bigg(\left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\frac{\partial X^3}{\partial x^1} - \frac{\partial X^1}{\partial x^3} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^3 \\
&\quad + \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3} \right) \mathrm{d}x^2 \wedge \mathrm{d}x^3 \Bigg) \\
&= \sharp \Bigg(\left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2} \right) \mathrm{d}x^3 + \left(\frac{\partial X^3}{\partial x^1} - \frac{\partial X^1}{\partial x^3} \right) \mathrm{d}x^2 + \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3} \right) \mathrm{d}x^1 \Bigg) \\
&= \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3} \right) \partial_1 + \left(\frac{\partial X^3}{\partial x^1} - \frac{\partial X^1}{\partial x^3} \right) \partial_2 + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2} \right) \partial_3.
\end{aligned}$$

In a similar way, one can define the divergence as

$$\mathrm{div}\; X := \star \mathrm{d} \star X^\flat.$$

This diagram can easily be extended to any oriented Riemannian 3-manifold (M, g).

References

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