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Dirichlet forms

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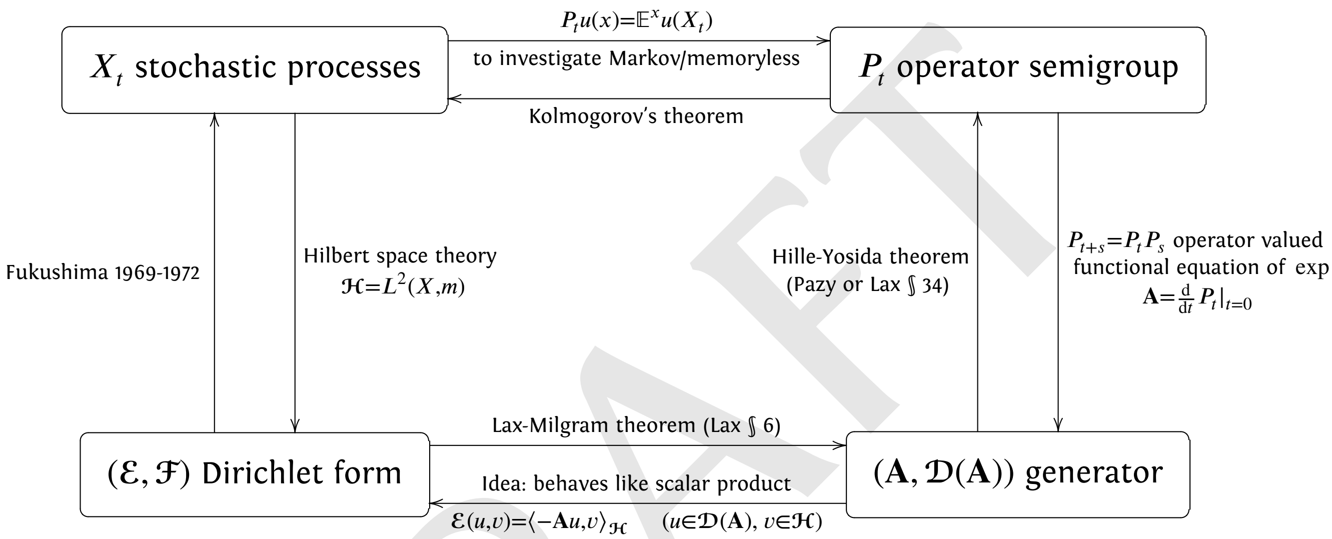
Chapter 0
Introduction

Goal understand BVP (boundary value problems) of PDEs / PsDEs via theory of stochastic processes



0.1 Theorem (Kolmogorv's theorem). *X_t is described by $\mathbb{P}^x\left(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n\right)$ for $0 = t_0 < \cdots < t_n$ and $B_1, \dots B_n$ Borel ($n \in \mathbb{N}$). Then there exists a stochastic process $(X_t)_{t \geqslant 0}$ or $(\mathbb{P}^x)_{x \in \cdot}$.*

Fukushima's idea:

$$
\begin{aligned}
P_t u(x) = \mathbb{E}^x u(X_t) &= \int u(y) \mathbb{P}^x \left(X_t \in \mathrm{d}y\right) \\
\implies \mathbb{P}^x \left(X_t \in B\right) &= P_t \mathbb{1}_B(x)
\end{aligned}
$$

Markov property gives the Chapman-Kolmogorov equation

$$
\mathbb{P}^x \left(X_{t+s} \in C, X_t \in B\right) = \int_{y \in B} \mathbb{P}^x(X_t \in \mathrm{d}y) \mathbb{P}^y(X_s \in C)
$$

Problem $P_t : L^2 \to L^2$, $\mathbb{1}_B \in L^2 \implies P_t \mathbb{1}_B \in L^2$ representative has null-set $N_{t,B}$ (over-countable many!). So we need control of $N_{t,B}$: build up with Chapman-Kolmogorov.

Chapter 1

Quadratic Forms and Bilinear Forms

Source: Oshima, de Gruyter or lectures from 1994, 1989.

Setting

- Hilbert space, mostly $\mathbb{R}$-Hilbert spaces: $\mathcal{H} = L^2(X, m)$,
- (X, d) a separable[1], locally compact[2] metric space with metric $d = d(x, y)$,
- m = Borel measures on $\mathcal{B}(X) = \sigma(d\text{-open sets})$ and Radon, i.e. $m(K) < \infty \ \forall K \subset X$ compact
- $\operatorname{spt} m = X$ «full support», i.e. $\forall U$ open: $m(U) > 0$ (to avoid $\sum$ Delta functions)[3]

1.1 Definition. $\mathcal{F} \subset L^2(X, m)$ dense linear subspace. A map $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a **(densely defined, real-valued) bilinear form**, if

$$\begin{aligned} u &\mapsto \mathcal{E}(u, v), \quad (v \in \mathcal{F}) \\ v &\mapsto \mathcal{E}(u, v), \quad (u \in \mathcal{F}) \end{aligned}$$

are $\mathbb{R}$-linear. $\mathcal{E}$ is **symmetric** if

$$\mathcal{E}(u, v) = \mathcal{E}(v, u) \quad (u, v \in \mathcal{F}).$$

Examples

- $\langle u, v \rangle = \langle u, v \rangle_{L^2} = \int_X u v \, \mathrm{d}m$,
- $\langle \mathbf{A} u, v \rangle_{L^2}$, where $\mathbf{A} : \mathcal{D}(\mathbf{A}) \subset L^2(X, m) \to L^2(X, m)$ linear operators.

Remark Why densely defined? This is usual needed for closure or (mostly) uniqueness questions.

1.2 Definition (Notations). a) $\mathcal{E}^s(u, v) := \frac{1}{2}(\mathcal{E}(u, v) + \mathcal{E}(v, u))$ symmetric part
$\curvearrowright \mathcal{E}(u, u) = \mathcal{E}^s(u, u) =: \mathcal{E}^s(u) :=: \mathcal{E}(u)$
b) $\mathcal{E}^a(u, v) := \frac{1}{2}(\mathcal{E}(u, v) - \mathcal{E}(v, u))$ antisymmetric part
$\curvearrowright \mathcal{E}^a(u, u) = 0$

---

[1] $\exists$ countable, dense subset
[2] $\forall x \exists$ a neighborhood $U(x)$, $\overline{U}(x)$ compact $\Longrightarrow$ finite dimensional space!
[3] Exercise: $m := \sum_{j=1}^{\infty} \alpha_j \delta_{x_j}$, $(x_j)_j \subset X$ dense

c) $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha \langle u, v\rangle_{L^2}$, $\alpha \in \mathbb{R}$, $\mathcal{E} = \mathcal{E}_0$ and $\mathcal{E}^s_\alpha, \mathcal{E}^a_\alpha$ etc.

1.3 Definition. The bilinear form $(\mathcal{E}, \mathcal{F})$ is **bounded from below (lower bounded)** if

($\mathcal{E}1$) $\exists \gamma \geqslant 0 : \mathcal{E}(u,u) \geqslant -\gamma \langle u,u\rangle_{L^2}$ or equivalently $\mathcal{E}_\gamma(u) \geqslant 0 \quad (u \in \mathcal{F})$

$(\mathcal{E}, \mathcal{F})$ satisfies the **sector condition (is sectorial)** with a **sector constant** κ ($\geqslant 1$) if there exists a $\kappa(\geqslant 1)$ and

($\mathcal{E}2$)
$$\begin{aligned}\mathcal{E}(u,v) &\leqslant \kappa \sqrt{\mathcal{E}_\gamma(u)}\sqrt{\mathcal{E}_\gamma(v)} \quad (u,v \in \mathcal{F}) \\ &\leqslant \kappa \sqrt{\mathcal{E}_\alpha(u)} \quad (u,v \in \mathcal{F}) \sqrt{\mathcal{E}_\alpha(v)} \quad (\alpha > \gamma),\end{aligned}$$

where the second equation is called **weak sector condition**. $(\mathcal{E}, \mathcal{F})$ is **closed** if in addition it holds

($\mathcal{E}3$) $(\mathcal{F}, \mathcal{E}^s_\alpha(\cdot,\cdot))$ is a Hilbert space $\alpha > \gamma$.

Remark You sometimes see the following notions: If $\gamma = 0$ in ($\mathcal{E}1$), $\mathcal{E}$ is **positive definite**, $\gamma < 0$, $\mathcal{E}$ is **coervice**.

1.4 Remark. Always $\alpha > \gamma$, $u, v \in \mathcal{F}$ (Exercise: consider $\alpha = \gamma$):

a) $\mathcal{E}^s_\alpha(u,v)$ is a scalar product, we always have

eq::1 (1) $|\mathcal{E}^s_\alpha(u,v)| \leqslant \sqrt{\mathcal{E}^s_\alpha(u)}\sqrt{\mathcal{E}^s_\alpha(v)}$

eq::1- (1′) $\sqrt{\mathcal{E}^s_\alpha(u \pm v)} \leqslant \sqrt{\mathcal{E}^s_\alpha(u)} + \sqrt{\mathcal{E}^s_\alpha(v)}$

eq::1-- (1″) $\left|\sqrt{\mathcal{E}^s_\alpha(u)} - \sqrt{\mathcal{E}^s_\alpha(v)}\right| \leqslant \sqrt{\mathcal{E}^s_\alpha(u \pm v)}$

b) $\alpha, \beta > \gamma$ we have $\mathcal{E}_\alpha(u) \asymp \mathcal{E}_\beta(u)$ for all $u \in \mathcal{F}$[4]

Proof. WLOG $\beta > \alpha > \gamma$.

$$\mathcal{E}_\alpha(u) = \mathcal{E}_\gamma(u) + \underbrace{(\alpha - \gamma)}_{\leqslant \beta - \gamma} \|u\|^2 \leqslant \mathcal{E}_\beta(u)$$

and the other direction follows from

$$\begin{aligned}\mathcal{E}_\alpha(u) &= \mathcal{E}_\gamma(v) + \frac{\alpha-\gamma}{\beta-\gamma}(\beta-\gamma)\|u\|^2 \\ &= \frac{\alpha-\gamma}{\beta-\gamma}\left(\frac{\beta-\gamma}{\alpha-\gamma}\mathcal{E}_\gamma(u) + (\beta-\gamma)\|u\|^2\right) \\ &\geqslant \frac{\alpha-\gamma}{\beta-\gamma}\mathcal{E}_\beta(u).\end{aligned}$$

$\curvearrowright$ all Hilbert spaces $(\mathcal{F}, \mathcal{E}^s_\alpha)_{\alpha>\gamma}$ are equivalent. ■

[4] $f \asymp g :\iff \exists c : cf(t) \leqslant g(t) \leqslant \frac{1}{c} f(t)$.

c) **Polarization indentities** $\alpha > \gamma$

$$\begin{aligned}\mathcal{E}^s_\alpha(u,v) &= \frac{1}{4}\left(\mathcal{E}^s_\alpha(u+v) - \mathcal{E}^s_\alpha(u-v)\right)\\ &= \frac{1}{2}\left(\mathcal{E}^s_\alpha(u+v) - \mathcal{E}^s_\alpha(u) - \mathcal{E}^s_\alpha(v)\right)\end{aligned}$$

d) Sector condition is a *control* on $\mathcal{E}^a$. Note that

$$\begin{aligned}|\mathcal{E}^a(u,v)| &= \frac{1}{2}\left|\mathcal{E}(u,v) + \mathcal{E}(v,u)\right|\\ &\leqslant \frac{1}{2}\left|\mathcal{E}(u,v)\right| + \frac{1}{2}\left|\mathcal{E}(v,u)\right| \qquad (2)\\ &\overset{(\mathcal{E}2)}{\leqslant} \kappa\sqrt{\mathcal{E}_\alpha(u)}\sqrt{\mathcal{E}_\alpha(v)},\end{aligned}$$

eq::2

mind ($\mathcal{E}2$) is «symmetric» in u and v on the right hand side. Moreover assume only (2), not yet ($\mathcal{E}2$), then:

$$\begin{aligned}|\mathcal{E}(u,v)| &\leqslant |\mathcal{E}^s(u,v)| + |\mathcal{E}^a(u,v)|\\ &\leqslant |\mathcal{E}^s_\alpha(u,v)| + |\mathcal{E}^a(u,v)| + \alpha\,|\langle u,v\rangle| \quad (\alpha > \gamma \geqslant 0)\\ &\overset{(a),\,(2)}{\leqslant} (\kappa+1)\sqrt{\mathcal{E}_\alpha(u)}\sqrt{\mathcal{E}_\alpha(v)} + \alpha\,\|u\|\,\|v\|\\ &\leqslant \left(\kappa + 1 + \frac{\alpha}{\alpha-\gamma}\right)\sqrt{\mathcal{E}_\alpha(u)}\sqrt{\mathcal{E}_\alpha(v)},\end{aligned}$$

where we used (last equation) $\mathcal{E}_\alpha(u) = \mathcal{E}_\gamma(u) + (\alpha-\gamma)\,\|u\| \geqslant \frac{\alpha-\gamma}{\alpha}\cdot\alpha\,\|u\|$.
Keep in mind $\alpha > \gamma$ gives coercive for $\mathcal{E}_\gamma$.

e) From ($\mathcal{E}2$) we get for $\alpha > \gamma$

$$|\mathcal{E}_\alpha(u,v)| \leqslant \left(\kappa + \frac{\alpha}{\alpha-\gamma}\right)\sqrt{\mathcal{E}_\alpha(u)}\sqrt{\mathcal{E}_\alpha(v)} \qquad (3)$$

eq::3

Hint: as in part d), difference is only α!.

f) closed: $(\mathcal{F}, \mathcal{E}^s_\alpha(\cdot,\cdot))$ is a Hilbert space. This is equivalent to saying:

$$(u_n)_n \subset \mathcal{F}, \quad \mathcal{E}_\alpha(u_n - u_m) \xrightarrow{n,m\to\infty} 0 \quad \mathcal{E}_\alpha\text{-Cauchy} \qquad (4)$$

eq::4

$$\Longrightarrow \exists u \in \mathcal{F} : \mathcal{E}_\alpha(u - u_n) \xrightarrow{n\to\infty} 0 \quad \mathcal{E}_\alpha\text{-convergence} \qquad (5)$$

eq::5

g) We may read the «$\alpha > \gamma$» in ($\mathcal{E}3$) either: $\exists\alpha > \gamma$ or $\forall\alpha > \gamma$. Indeed: b).

h) ($\mathcal{E}2$) with $\alpha > \gamma$ is «weak» sector condition, $\alpha = \gamma$ strong sector condition.

Problem is usually ($\mathcal{E}3$). Hard to verify in concrete cases ($\longrightarrow$ Sobolev spaces!). Way out (partially) is notion of **closabilty**.

1.5 Definition. A lower bounded, sectorial, bilinear form $(\mathcal{E},\mathcal{F})$ is **closable** if for some (then for all) $\alpha > \gamma$ we have

$$(u_n)_n \subset \mathcal{F}, \quad \mathcal{E}_\alpha(u_n - u_m) \xrightarrow{n,m\to\infty} 0, \quad \left\|u_n\right\|_{L^2} \to 0$$
$$\Longrightarrow \mathcal{E}_\alpha(u_n) \xrightarrow{n\to\infty} 0 \qquad (6) \quad \boxed{\texttt{eq::6}}$$

1.6 Definition. A bilinear form $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ extends $(\mathcal{E},\mathcal{F})$, if

$$\mathcal{F} \subset \overline{\mathcal{F}} \quad \text{and} \quad \overline{\mathcal{E}}(u,v) = \mathcal{E}(u,v) \quad (u,v \in \mathcal{F}) \qquad (7) \quad \boxed{\texttt{eq::7}}$$

We write: $\overline{\mathcal{E}} \supset \mathcal{E}$.

1.7 Lemma. *A lower bounded, sectorial, bilinear form $(\mathcal{E},\mathcal{F})$ has a closed extension $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ iff $(\mathcal{E},\mathcal{F})$ is closable.*

Proof. $\Longleftarrow$ Let $(\mathcal{E},\mathcal{F})$ be closable. Fix $\alpha > \gamma$.

$$\mathcal{L} = \left\{(u_n)_n \subset \mathcal{F} \,:\, (u_n)_n \text{ is } \mathcal{E}_\alpha\text{-Cauchy}\right\},$$
$$(u_n)_n \sim (u_n')_n \;:\Longleftrightarrow \lim_n \mathcal{E}_\alpha(u_n - u_n') = 0$$

Claim $\overline{\mathcal{F}} := \mathcal{L}/\!\sim$, $\overline{\mathcal{E}}(u,v) = \lim_n \mathcal{E}(u,v) \Longrightarrow (\overline{\mathcal{E}},\mathcal{F})$ does the job.

1° $\lim_n \mathcal{E}_\alpha(u)$ exists $\forall (u_n) \in \mathcal{L}$ ($\alpha > \gamma$ fixed) and use the lower triangle inequality (1′′)

$$\left|\sqrt{\mathcal{E}_\alpha(u_n)} - \sqrt{\mathcal{E}_\alpha(u_m)}\right| \leqslant \sqrt{\mathcal{E}_\alpha(u_n - u_m)} \xrightarrow{n,m\uparrow\infty} 0$$

$\curvearrowright \left(\mathcal{E}_\alpha(u_n)\right)_n$ Cauchy in $\mathbb{R}$.

2° $\lim_n \mathcal{E}_\alpha(u_n, v_n)$ exists $\forall (u_n),(v_n) \in \mathcal{L}$

$$\left|\mathcal{E}(u_n,v_n) - \mathcal{E}(u_m,v_m)\right| = \left|\,\mathcal{E}(u_n, v_n - v_m) + \mathcal{E}(u_n - u_m, v_m)\right|$$
$$\overset{\text{w-sec}}{\leqslant} \kappa \underbrace{\mathcal{E}_\alpha^{1/2}(u_n)}_{\text{bdd } 1^\circ} \underbrace{\mathcal{E}_\alpha^{1/2}(v_n - v_m)}_{\to 0} + \kappa \underbrace{\mathcal{E}_\alpha^{1/2}(u_n - u_m)}_{\to 0} \underbrace{\mathcal{E}_\alpha^{1/2}(v_m)}_{\text{bdd } 1^\circ}$$
$$\xrightarrow{m,n\uparrow 0},$$

so it is again Cauchy. Exercise: show that $\left(\langle u_n, v_n\rangle_{L^2}\right)_n$ also Cauchy. Hint: $\alpha > \gamma$, use $\mathcal{E}_\alpha(u) \geqslant 0$ and $(u_n) \in \mathcal{L}$ is also $\|\cdot\|_{L^2}$ –Cauchy, use estimate at the end of remark 1.4 d): $\mathcal{E}_\alpha(u) \geqslant (\alpha-\gamma)\,\|u\|_{L^2}$.

3° Well definedness ($\lim_n$ is independent of the sequence). Same argument as in 2°: $(u_n) \sim (u_n')$, $(v_n) \sim (v_n')$. Replace in 2° u_m with u_n' and v_m with v_n', then $\mathcal{E}_\alpha(u_n - u_n') \to 0$ by equivalence, $\mathcal{E}_\alpha(u_n)$ is bounded since (some!) lim exists.

4° $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ is closed. Pick a sequence $(w_n) \subset \overline{\mathcal{F}}$. Each w_n has an approximating sequence $(w_{n,k})_k \subset \mathcal{F}$. So

$$\forall \varepsilon > 0 \, \forall n \in \mathbb{N} \, \exists N_{\varepsilon,n} \forall k \geqslant N_{\varepsilon,n} : \overline{\mathcal{E}}_\alpha(w_{n,k} - w_n) \leqslant \varepsilon.$$

If $(w_n)_n$ is $\overline{\mathcal{E}}_\alpha$-Cauchy, we get

$$\begin{aligned}\left(\mathcal{E}(w_{n,k} - w_{m,l})\right)^{1/2} &= \left(\overline{\mathcal{E}}(w_{n,k} - w_{m,l})\right)^{1/2} \\ &\leqslant \left(\mathcal{E}(w_{n,k} - w_n)\right)^{1/2} + \left(\mathcal{E}(w_n - w_m)\right)^{1/2} + \left(\mathcal{E}(w_m - w_{m,l})\right)^{1/2} \\ &\leqslant 3\varepsilon,\end{aligned}$$

if $n, m \geqslant N_\varepsilon$, from $\overline{\mathcal{E}}_\alpha$-Cauchy, and $k \geqslant N(n, \varepsilon)$. Use diagonal sequence $(w_{n,N(n,\frac{1}{n})})_{n\in\mathbb{N}} \in \mathcal{L}$, i.e. defined an element w in $\overline{\mathcal{F}}$ (by construction) and $\overline{\mathcal{E}}_\alpha(w_{n,N(n,\frac{1}{n})} - w) \to 0$.

5° $\overline{\mathcal{E}}_\alpha$ extends $\mathcal{E}_\alpha$. Trivial: $u \in \mathcal{F}$, then $(u, u, u, dots) \in \mathcal{L}$, i.e. $u \in \overline{\mathcal{F}}$.

$\Longrightarrow$ Let $\overline{\mathcal{E}} \supset \mathcal{E}$ be a closed extension. Let $(u_n)_n \subset \mathcal{F} \subset \overline{\mathcal{F}}$, $\overline{\mathcal{E}}_\alpha(u_n - u_m) = \mathcal{E}_\alpha(u_n - u_m) \to 0$ (Cauchy) and $\|u_n\|_n \to 0$. To show: $\mathcal{E}_\alpha(u_n) \to 0$. Use $\overline{\mathcal{E}}_\alpha$ is closed, i.e. $\overline{\mathcal{E}}_\alpha(u_n - u) \xrightarrow{n\uparrow\infty} 0$ for some $u \in \overline{\mathcal{F}}$. To show: $u = 0$. Since L^2 is closed: $\|u_n\|_{L^2} = \|u_n - 0\|_{L^2} \xrightarrow{n\uparrow\infty} 0$ by assumption, so $u = 0$.

■

Next aim Structure of such forms

Recall Linear algebra q = symmetric bilinear form on a n-dimensional vector space V, with basis $b_1, \dots, b_n$, then

$$q(x, y) = x^t \mathbf{A} y = \langle x, \mathbf{A} y \rangle,$$

$\langle \cdot, \cdot \rangle$ the scalar product on V. $\mathbf{A} = (a_{ij})$, $a_{ij} = q(b_i, b_j)$ symmetric «structural matrix».

Idea «$\mathcal{E}_\alpha(u, v) = \langle u, \mathbf{A} v \rangle_{L^2(X,m)}$», where $\mathbf{A}$ is a linear operator

Functional analysis Lax-Milgram theorem, but we are not symmetric.

qfbf-18 **1.8 Theorem** (Stampacchia 1964, non-linear version, extends Lax-Milgram). *$\mathcal{E}$ is closed ($\Longrightarrow$ lower bounded, sectorial) bilinear form on $\mathcal{F}$, $\Gamma \subset \mathcal{F}$, $\Gamma \neq \emptyset$ closed convex subset. $\mathcal{J}$ is a continuous (w.r.t $\mathcal{E}_\alpha$, $\alpha > \gamma$) linear functional on $\mathcal{F}$:*

$$\Longrightarrow \exists!\, v \in \Gamma \forall w \in \Gamma : \mathcal{E}_\alpha(v, w - v) \geqslant \mathcal{J}(w - v) \tag{7}$$ eq::7

(7) is usually called «variational inequality».

1.9 Remark. Γ = closed subspace. Then (7) $\iff$

$$\exists!\, v \in \Gamma\, \forall w \in \Gamma : \mathcal{E}_\alpha(v, w) \geqslant \mathcal{J}(w) \tag{7'}$$ eq::7-

Proof of theorem 1.8. 1° **Stability** Assume u_1, u_2 solve the problems: $\forall w \in \Gamma$

$$\mathcal{E}_\alpha(u_1, w - u_1) \geqslant \mathcal{J}_1(w - u_1)$$
$$\mathcal{E}_\alpha(u_2, w - u_2) \geqslant \mathcal{J}_2(w - u_2)$$

Pick $w = u_2$ and $w = u_1$, respectively. Then add inequalities

$$\begin{aligned} \mathcal{E}_\alpha(u_2 - u_1, u_1 - u_2) &\geqslant \mathcal{J}_1(u_2 - u_1) + \mathcal{J}_2(u_1 - u_2) \\ \mathcal{E}_\alpha(u_1 - u_2) &\leqslant \mathcal{J}_1(u_2 - u_1) + \mathcal{J}_2(u_1 - u_2) \\ &\leqslant \|\mathcal{J}_1 - \mathcal{J}_2\|\, \mathcal{E}_\alpha^{1/2}(u_1 - u_2) \end{aligned} \tag{8}$$ eq::8

So $\mathcal{E}_\alpha$-continuity of $\mathcal{J}$ just means:

$$\|\mathcal{J}(u)\| \leqslant \|\mathcal{J}\|\, \mathcal{E}_\alpha^{1/2}(u)$$

2° **Uniqueness** of the original problem: know $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}$, so $u_1 = u_1$ by 1°, i.e. only *one* solution possible.

3° **Existence of $\mathcal{E}$ is symmetric** Auxiliary function

$$\begin{aligned} \mathfrak{I}(v) &= \mathcal{E}_\alpha(v) - 2\mathcal{J}(v) \quad (v \in \Gamma) \\ d &= \inf_{v \in \Gamma} \mathfrak{I}(v) \end{aligned} \tag{9}$$ eq::9

By continuity, we get $|\mathcal{J}(v)| \leqslant \|\mathcal{J}\|\, \mathcal{E}_\alpha^{1/2}(v)$ and

$$|\mathfrak{I}(v)| \geqslant \underbrace{\mathcal{E}_\alpha(v) - 2\,\|\mathcal{J}\|\, \mathcal{E}_\alpha^{1/2}(v) + \|\mathcal{J}\|^2}_{\text{perfect square, } \geqslant 0} - \|\mathcal{J}\|^2 \geqslant -\|\mathcal{J}\|^2,$$

$$\overset{\forall v \in \Gamma}{\Longrightarrow} d \geqslant -\|\mathcal{J}\|^2 > -\infty,$$

so the inf can be attained. So $\exists (v_n)_n \subset \Gamma : \mathfrak{I}(v_n) \xrightarrow{n \uparrow \infty} d$.

Aim now v_n converges. Uses parallelogram indentity for $\mathcal{E}_\alpha(\cdot, \cdot)$:

$$\begin{aligned} \mathcal{E}_\alpha(v_n - v_m) + \mathcal{E}_\alpha(v_n + v_m) &= 2\mathcal{E}_\alpha(v_m) + 2\mathcal{E}_\alpha(v_m) \\ \mathcal{E}_\alpha(v_n - v_m) &= -4\mathcal{E}_\alpha \underbrace{\left(\frac{v_n + v_m}{2}\right)}_{\in \Gamma, \text{ convex}} + 2\mathcal{E}_\alpha(v_m) + 2\mathcal{E}_\alpha(v_m) \\ &= \underbrace{2\mathfrak{I}(v_n)}_{\to 2d} + \underbrace{2\mathfrak{I}(v_m)}_{\to 2d} - 4 \underbrace{\mathfrak{I}\left(\frac{v_n + v_m}{2}\right)}_{\geqslant d} \quad \text{(mind } \mathcal{J} \text{ is linear)} \end{aligned}$$

$$\limsup_{n,m \to \infty} \mathcal{E}_\alpha(v_n - v_m) \leqslant 4d - 4d = 0,$$

so $\limsup = 0 \Longrightarrow \lim = 0$ (since the sequence was positive) $\Longrightarrow (v_n)_n$ is Cauchy.
$\Longrightarrow v = \mathcal{E}_\alpha - \lim_{n\to\infty} v_n$ exists
$\Longrightarrow v \in \Gamma$ since Γ is closed $\Longrightarrow$ inf is attained.

Finally $\forall \varepsilon \in (0,1)\, \forall w \in \Gamma$

$$\begin{aligned} 0 &\leqslant \mathfrak{I}(v + \varepsilon(w-v)) - \mathfrak{I}(v) \\ &= 2\varepsilon \mathcal{E}_\alpha(v, w-v) - 2\varepsilon \mathcal{J}(w-v) + \varepsilon^2 \mathcal{E}_\alpha(w-v) \end{aligned}$$

Now divide by ε, let $\varepsilon \downarrow 0$, rearrange and get (7).

4° **Non-symmetric part** Idea: it is a perturbation of symmetric part. Need auxiliary form:

$$q_t(u,v) = \mathcal{E}^s_\alpha(u,v) + t\mathcal{E}^a_\alpha(u,v) \quad (0 \leqslant t \leqslant 1)$$

Assume the assertion of the theorem holds for *some* $\tau \in [0,1)$,[5] i.e.

$$\exists!\, v \in \Gamma\, \forall w \in \Gamma : \, q_\tau(v, w-v) \geqslant \mathcal{J}(w-v).$$

$\tau = 0$ is just 3°. We show: can replace τ by $t \in [\tau, \tau + \star)$. Let $u \in \mathcal{F}$, $t > \tau$. Then $v \mapsto \mathcal{J}(v) - (t-\tau)\mathcal{E}^a_\alpha(u,v)$ is a linear functional (u fixed!), it is $q_\varepsilon/\mathcal{E}_\alpha$-cts:

$$\begin{aligned} |\mathcal{J}(v) - (t-\tau)\mathcal{E}^a_\alpha(u,v)| &\overset{(3)}{\leqslant} \|\mathcal{J}\|\, \mathcal{E}^{1/2}_\alpha(v) + (t-\tau)\left(\kappa + \frac{\alpha}{\alpha+\gamma}\right)\sqrt{\mathcal{E}_\alpha(u)}\sqrt{\mathcal{E}_\alpha(v)} \\ &= \left[\|\mathcal{J}\| + (t-\tau)\left(\kappa + \frac{\alpha}{\alpha+\gamma}\right)\sqrt{\mathcal{E}_\alpha(u)}\right] \underbrace{\sqrt{\mathcal{E}_\alpha(v)}}_{=\sqrt{q_t(v)}}. \end{aligned}$$

Use this new linear functional instead of our $\mathcal{J}$ in the «induction assumption»: $\exists!\, v = Tu \in \Gamma : \forall w \in \Gamma$

$$q_\tau(Tu, w - Tu) \geqslant \mathcal{J}(w - Tu) - (t-\tau)\mathcal{E}^a_\alpha(u, w-Tu),$$

study $u \mapsto Tu$ (in general not linear). Take $u = u_1$, $u =_2$ and step 1°, $\mathcal{J}_j(x) = \mathcal{J}(x) - (t-\tau)\mathcal{E}^a_\alpha(u_j, x)$, then by (8)

$$\begin{aligned} q_\tau(Tu_1 - Tu_2) &\overset{(8)}{\leqslant} (t-\tau)\mathcal{E}^a_\alpha(u_1 - u_2, Tu_1 - Tu_2) \\ &\overset{(3)}{\leqslant} (t-\tau)\left(\kappa + \frac{\alpha}{\alpha-\gamma}\right)\sqrt{\mathcal{E}_\alpha(u_1-u_2)}\sqrt{\mathcal{E}_\alpha(Tu_1 - Tu_2)}, \end{aligned}$$

[5] Mind: $q_\varepsilon(u,u) = \mathcal{E}_\alpha(u,u) = \mathcal{E}^s_\alpha(u,u)$, so q_ε-cts = $\mathcal{E}_\alpha$-cts.

$\Longrightarrow \sqrt{\mathcal{E}_\alpha(Tu_1 - Tu_2)} \leqslant (t-\tau)\left(\kappa + \frac{\alpha}{\alpha-\gamma}\right)\sqrt{\mathcal{E}_\alpha(u_1-u_2)}$, pick t such that all < 1. So T is an $\mathcal{E}_\alpha$-contraction and by Banach $\exists$ fixed point $Tv = v$ and so

$$q_\tau(v, w-v) \geqslant \mathcal{J}(w-v) - (t-\tau)\mathcal{E}^a_\alpha(v, w-v)$$
$$q_t(v, w-v) \geqslant \mathcal{J}(w-v) + \text{ iterate.}$$

■

1.10 Example (mostly symmetric, $\gamma = 0$). a) *the* closed form: Let $D \subset \mathbb{R}^d$ an open domain

$$\mathcal{E}(u,v) = \frac{1}{2}\int_D \nabla u \cdot \nabla v \,\mathrm{d}x \quad (u,v \in C_c^\infty(D)) \tag{10}$$ eq::10

$(\mathcal{E}_1)$, $(\mathcal{E}_2)$ clear. As usual (with all examples!) «closed» is problematic. C_c^∞ is too small. $\mathcal{E}$ only controls at best 2 derivatives, converges in $L^2(m)$, but does not preserve continuity. So use *Sobolev spaces*

$$\mathcal{F} = W^1(D) = \left\{ u \in \mathcal{S}' : u \in L^2(D, \mathrm{d}x) \text{ and } \frac{\partial u}{\partial x_i} \in L^2(D, \mathrm{d}x) \right\}, \tag{11}$$ eq::11

where $\frac{\partial u}{\partial xi}$ is a weak or distributional derivative. Idea is

$$\underbrace{\left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle_{L^2}}_{\text{this is a lin. fn.al def. by RHS}} := \int_D \frac{\partial u}{\partial x_i}\varphi \mathrm{d}x = -\int_D u \frac{\partial}{\partial x_i}\varphi \mathrm{d}x \quad (\varphi \in C_c^\infty(D), u \in L^2(D)),$$

but if $\varphi \in C_c^\infty(D)$ we have zero boundary. If $\frac{\partial u}{\partial x_i}$ exists classically and is $\in L^2(D)$ we say that the linear functional **represented** by $\frac{\partial u}{\partial x_i}$. In general

$$\frac{\partial u}{\partial x_i} = \left\langle \frac{\partial u}{\partial x_i}, \cdot \right\rangle_{L^2} \in \mathcal{D}'(D), \frac{\partial u}{\partial x_i} = {}_{\mathcal{D}'}\left\langle \frac{\partial u}{\partial x_i}, \cdot \right\rangle_{\mathcal{D}} \in \mathcal{D}'(D),$$

and note the old (french) notation $\mathcal{D}(D) = C_c^\infty(D)$, $\mathcal{D}'(D) = \left(C_c^\infty(D)\right)^*$.

Prove of closedness. Let $(u_n) \subset \mathcal{F}$, $\mathcal{E}_1$-Cauchy, i.e. $\mathcal{E}_1(u_n - u_m) \xrightarrow{n,m\uparrow\infty} 0$, meaning

$$\left\|u_n - u_m\right\|_{L^2(D,\mathrm{d}x)} \xrightarrow{n,m\uparrow\infty} 0 \underset{\text{complete}}{\overset{L^2}{\Longrightarrow}} \exists u \in L^2(D,\mathrm{d}x) : u_n \xrightarrow{L^2} u \tag{$*$}$$

$$\left\|\nabla u_n - \nabla u_m\right\| \xrightarrow{n,m\uparrow\infty} 0 \Longrightarrow \exists v \in L^2(D,\mathrm{d}x) : \nabla u_n \xrightarrow{L^2} v. \tag{$**$}$$

■

Problem $v = \nabla u$?

Take $\forall \varphi \in C_c^\infty(D)$

$$\langle v, \varphi \rangle_{L^2} \overset{L^2}{\longleftarrow} \langle \nabla u_n, \varphi \rangle_{L^2} = -\langle u_n, \nabla\varphi \rangle_{L^2} \xrightarrow[n\uparrow\infty]{*} -\langle u, \nabla\varphi \rangle$$

$\Longrightarrow$ v has weak derivative ∇u

$\Longrightarrow$ $u \in \mathcal{F}$ and ∇u exists and $v = \nabla u$

b) how big or small is $C_c^\infty(D)$? Very small!

$$\overline{C_c^\infty}^{\mathcal{E}_1(\cdot)} \neq W^1(D) = W_0^1(D)$$

Fact If ∂D regular

$$W_0^1(D) = \left\{ u \in W^1(D) \,:\, u\,|_{\partial D} = 0 \right\},$$

where we use the trace operator

$$\begin{aligned} \gamma \,:\, C_c^\infty(\overline{D}) &\to L^2(\partial D) \\ u &\mapsto u\,|_{\partial D}, \end{aligned}$$

and show γ is continuous.[6]

c) **Integration by parts**

$$\mathcal{E}(u,v) = \underbrace{\frac{1}{2}\int_D \nabla u \cdot \nabla v \mathrm{d}x}_{\text{defined on } \mathcal{F}, u, v, \nabla u, \nabla v \in L^2} = \underbrace{\frac{1}{2}\int_D -\Delta u \cdot v \mathrm{d}x}_{\text{defined on } u, \nabla u, \Delta u \in L^2, v \in L^2},$$

but the left side is a form $\mathcal{E}(u,v), \mathcal{F}$ and the righthand side is a generator $L = \nabla$, $\langle -Lu, v \rangle$ for $u \in \mathcal{D}(L), v \in L^2$. On the RHS we have

$$u \in W_0^2(D) = \left\{ u \in \mathcal{D}' \,:\, u \in L^2(D),\, \nabla u \in L^2(D),\, \Delta u \in L^2(D),\, u\,|_{\partial D} = 0,\, \nabla u\,|_{\partial D} = 0 \right\},$$

is the domain of the Dirichlet-Laplace operator

d) Structure of most general symmetric closed forms on $\mathbb{R}^d$ which are interesting to us[7]

$$\begin{aligned} \mathcal{E}(u,v) = &\sum_{i,j=1}^d \int_D \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} v_{ij}(\mathrm{d}x) + \int_{D\times D\setminus \mathrm{diag}} (u(x)-u(y))\,(v(x)-v(y))\,\mathcal{J}(\mathrm{d}x,\mathrm{d}y) \\ &+ \int_D u(x)\,v(x) k(\mathrm{d}x) \qquad (12) \\ = &\text{ 2nd ordner term } + \alpha\text{th order term, } 0 < \alpha < 2 \, + \text{ 0th order term ,} \end{aligned}$$

eq:12

[6]Source: any book of Sobolev spaces, e.g. Adams: Sobolev spaces.
[7]giving semigroups and processes.

here $\nu_{ij} = \nu_{ji}$, k = radon measure on D, $\mathfrak{J}$ = Radon on $D \times D \setminus \text{diag}$ and

- $\displaystyle\int_{K\times K\setminus\text{diag}} |x-y|^2\, \mathfrak{J}(\mathrm{d}x, \mathrm{d}y) < \infty \quad (K \subset D \text{ compact})$
- $\mathfrak{J}(K \times (D \setminus U)) < \infty \quad (K \subset U \subset D,\ U \text{ open},\ \overline{U} \subset D \text{ compact})$

Exercise above should be equivalent to $\int_{D\times D\setminus\text{diag}} \frac{|x-y|^2}{1+|x+y|^2} \mathfrak{J}(\mathrm{d}x, \mathrm{d}y) < \infty$.

WLOG $\mathfrak{J}(\mathrm{d}x, \mathrm{d}y) = \mathfrak{J}(\mathrm{d}y, \mathrm{d}x)$ else: $\tilde{\mathfrak{J}}(\mathrm{d}x, \mathrm{d}y) := \frac{1}{2}\left(\mathfrak{J}(\mathrm{d}x, \mathrm{d}y) + \mathfrak{J}(\mathrm{d}y, \mathrm{d}x)\right)$

further $\displaystyle\sum_{i,j=1}^{d} \xi_i \xi_j \nu_{ij}(K) \geqslant 0$ (K compact, $\xi_i, \xi_j \in \mathbb{R}^d$), ensures positivity (i.e. $\gamma = 0$ in ($\mathcal{E}1$)

clear $\mathcal{E}$ is symmetric, bilinear, $\gamma = 0$

easy $\mathcal{E}$ defined for $u, v \in C_c^\infty(D)$

hard $\mathcal{E}$ closable on $C_c^\infty(D)$ needs more assumptions.

show $\mathcal{E}$ makes sense for $u, v \in C_c^\infty$: 1st, 3rd term on rhs of 12. 2nd termn: split $\mathfrak{J}$-integral:

$$\int \ldots \mathfrak{J}(\mathrm{d}x, \mathrm{d}y) = \int_{U\times U\setminus\text{diag}} + \int_{U\times(D\setminus U)} + \int_{(D\setminus U)\times U} + \int_{(D\setminus U)\times(D\setminus U)\setminus\text{diag}},$$

$U \subset D$ so that $\operatorname{spt} u$, $\operatorname{spt} v \subset U$, $\overline{U}$ compact. By the mean value theorem

$$|u(x) - u(y)| \leqslant c\,|x-y|$$

in the 1st integral, next

$$\begin{aligned}\int_{U\times(D\setminus U)} (u(x)-u(y))\,(v(x)-v(y))\,\mathfrak{J}(\mathrm{d}x, \mathrm{d}y) &= \int_{U\times(D\setminus U)} u(x)\,v(x)\mathfrak{J}(\mathrm{d}x, \mathrm{d}y)\\ &\leqslant \|u\|_\infty \|v\|_\infty\, \mathfrak{J}(K \times (D \setminus U)),\end{aligned}$$

$K = \operatorname{spt} u \cup \operatorname{spt} v$ is compact and $\subset U$, where U is open. Fourth

$$\int_{(D\setminus U)\times(D\setminus U)\setminus\text{diag}} \cdots = 0,$$

as $\operatorname{spt} u, \operatorname{spt} v \subset U$.

$\gamma = 0$ and 1st integral on the rhs of 12

$$\sum_{i,j=1}^{d} \int_D \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \nu_{ij}(\mathrm{d}x) \overset{!}{\geqslant} 0,$$

have $\displaystyle\sum_{i,j=0}^{d} \int \xi_i \xi_j \nu_{ij}(K) \geqslant 0 \quad (K \text{ compact})$

simplex idea for have to $\overset{!}{\geqslant}$

C_k so small, that

$$\frac{\partial u(x)}{\partial x_i} \approx \frac{\partial u(\eta_k)}{\partial x_i}, \text{e.g. } \eta_k = \text{ center of } C_k$$
$$\Longrightarrow \int_D \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \mathrm{d}\nu_{ij} \approx \sum_{k=1}^{l} \int_{C_k} \frac{\partial u(\eta_k)}{\partial x_i}\frac{\partial u(\eta_k)}{\partial x_j} \mathrm{d}\nu_{ij},$$

Remark make «≈» rigorous by Lebesgue's Differentiation

Theorem $\forall u \in L^1(\mu)$. Then (w.r.t μ) in x

$$\lim_{C \downarrow \{x\}} \frac{1}{\mu(C)} \int_C f(y)\mu(\mathrm{d}y) = f(x) \quad \text{a.e..}$$

where C is nicely shrinking, e.g. $B(x, r)$, $r \downarrow 0$.

e) $I = (a, b) \subset \mathbb{R}$, $a < b$, $a, b \in \overline{\mathbb{R}}$,

$$\mathbb{D}(u, v) = \int_I u'(x) v'(x) \mathrm{d}x,$$
$$\mathfrak{F}^R = \left\{ u \in L^2(I, m) \cap L^2(I, k),\, u \text{ is absolutely continuous and } \mathbb{D}(u, v) < \infty \right\}.$$

base space: $L^2(I, m)$, m Radon, spt $m = I$,

$$\mathcal{E}(u, v) = \frac{1}{2}\mathbb{D}(u, v) + \int_I u \cdot v \mathrm{d}k,\ \mathfrak{F} = \mathfrak{F}^R,$$
$$\mathcal{E}_1(u, v) = \frac{1}{2}\mathbb{D}(u, v) + \int_I u \cdot v \mathrm{d}k + \int_I u \cdot v \mathrm{d}m$$

$(\mathcal{E}, \mathfrak{F})$ is closed.

Proof of closedness.. $(u_n) \subset \mathfrak{F}^R$, $\mathcal{E}_1$-Cauchy. Then

$$u_n' \xrightarrow{L^2(\mathrm{d}x)} f \in L^2(\mathrm{d}x),$$
$$u_n \xrightarrow{L^2(m) \cap L^2(k)} u \in L^2(m) \cap L^2(k).$$

Aim u' exists and $= f$. $\forall x, y \in I$ $\forall u \in \mathfrak{F}^R$

$$|u(x) - u(y)|^2 = \left| \int_x^y u'(t) \mathrm{d}t \right|^2 \leqslant |x - y|^2 \, \mathbb{D}(u)$$

$\Longrightarrow \exists n_j,\ u_{n_j} \to \tilde{u}$ cts, locally uniform convergence[8]
$\Longrightarrow u = \tilde{u}$ $(m + k)$-a.e. and $\forall \varphi \in C_c^\infty(I)$:

$$\int_I f \, \varphi \mathrm{d}x = \lim_{n_j \to \infty} \int_I u'_{n_j} \, \varphi \mathrm{d}x = -\lim_{n_j} \int_I u_{n_j} \, \varphi' \mathrm{d}x = -\int \tilde{u} \, \varphi' \mathrm{d}x,$$

i.e. $\tilde{u}$ is absolutely continuous and $\tilde{u} = f$, i.e. $\tilde{u}' \in \mathfrak{F}$. ■

[8] exercise, come from the bound above, usual covering and 3ε trick

Chapter 2

Semigroups, Resolvents, Generators

aim Form $\oplus$ extra properties = Dirichlet form $\xrightarrow{\text{Stampacchia}}$ Resolvent $\to$ semigroup Markovian

aim in § 2 Hille-Yosida therem

setting $(\mathcal{B}, \|\cdot\|)$ Banach space over $\mathbb{R}$, all «operators» are linear

2.1 Definition. A family of bounded linear operators[1] $(P_t)_{t\geqslant 0}$, $P_t : \mathcal{B} \to \mathcal{B}$ is a (C_0)-**semigroup** or **strongly continuous semigroup** of **contractions** if

$$P_0 = \mathrm{id}, \quad P_{t+s} = P_t \circ P_s (= P_s P_t) \tag{1}$$

$$\|P_t u\| \leqslant \|u\| \quad \forall u \iota \mathcal{B} \tag{2}$$

$$\lim_{t\to 0} \|P_t u - u\| = 0 \tag{3}$$

$$\lim_{s\to t} \|P_t u - P_s u\| = 0 \quad \forall s, t \tag{3'}$$

Idea (1) - (3) is an «operator valued» functional equation of the type:

$$\varphi(s+t) = \varphi(s)\varphi(t), \quad \varphi(0) = 1,$$

which has a solution $\varphi(t) = e^{t\,a}$, $a \in \mathbb{R}$ suitable, $a = \varphi'(0+)$, ok if $\varphi : \mathbb{R}_+ \xrightarrow{\text{cts}} \mathbb{R}$.
In operator case: a is a linear operator, but

$$\exp(a) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{a^n}{n!} = \sum_{n=0}^{\infty} \frac{a \circ \ldots \circ a}{n!},$$

converges if a is a bounded operator. Problem if a is not bounded in $\mathcal{B}$! Typical example is $\Delta : \mathcal{D}(\Delta) \subset C \to C$.

2.2 Definition. $(P_t)_{t\geqslant 0}$ (C_0)-contraction semigroup. The linear operator defined by

$$\mathcal{D}(\mathbf{A}) = \left\{ u \in \mathcal{B} : \lim_{t\downarrow 0} \frac{P_t u - u}{t} \text{exists (strongly) in } \mathcal{B} \right\} \tag{4}$$

$$\forall u \in \mathcal{D}(\mathbf{A}) : \mathbf{A}u := \lim_{t\downarrow 0} \frac{P_t u - u}{t} = \left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} P_t u \tag{2.1}$$

[1] Recall, bounded means $\exists c\, \forall x \in \mathcal{B} : \|Tx\| \leqslant c\,\|x\| \iff$ cts (at 0)

is called the **(infinitesimal) generator** of $(P_t)_{t\geqslant 0}$.

Study relation $P_t \longleftrightarrow \mathbf{A}$. Need a good notion of integrals of $\mathcal{B}$-valued integrands. Let $u : [a,b] \to \mathcal{B}$, $t \mapsto u(t)$ is continuous. We can define

$$\int_a^b u(s)\mathrm{d}s \overset{\text{def}}{=} \lim \text{Riemann-Sums with convergence in } (\mathcal{B}, \|\cdot\|).$$

This object has all good properties of a Riemann-sums as we know it from $\mathbb{R}$.

2.3 Lemma. *Let $u : [a,b] \to \mathcal{B}$ be continuous.*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_a^{a+\varepsilon} u(t)\mathrm{d}t = u(a) \quad \textit{strong/}\mathcal{B}\textit{-convergence.}$$

Proof.

$$\begin{aligned}\left\|\frac{1}{\varepsilon}\int_a^{a+\varepsilon} u(t)\mathrm{d}t - u(a)\right\| &= \left\|\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_a^{a+\varepsilon}(u(t)-u(a))\mathrm{d}t\right\| \\ &\leqslant \frac{1}{\varepsilon}\int_a^{a+\varepsilon} \underbrace{\|u(t)-u(a)\|}_{\text{cts in } t}\,\mathrm{d}t \\ &\leqslant \sup_{a\leqslant t\leqslant a+\varepsilon}\|u(t)-u(a)\| \xrightarrow[\text{uniformly cts}]{\varepsilon\downarrow 0} 0.\end{aligned}$$

■

srg-24 **2.4 Lemma.** $(P_t)_{t\geqslant 0}$ *(C_0)-contraction semigroup.*

$$\forall u \in \mathcal{D}(\mathbf{A})\,\forall t \geqslant 0 : \frac{\mathrm{d}}{\mathrm{d}t}P_t u = \mathbf{A}P_t u = P_t\mathbf{A}u \tag{6}$$ eq::srg6

$$\forall u \in \mathcal{B} : \int_0^t P_s u\mathrm{d}s \in \mathcal{D}(\mathbf{A}) \tag{7}$$

$$\forall u \in \mathcal{D}(\mathbf{A}) : P_t u - u = \underbrace{\mathbf{A}\int_0^t P_s u s}_{ok\ \forall u\in\mathcal{B}} = \int_0^t \mathbf{A}P_s u\mathrm{d}s \tag{8}$$

Attention $\mathbf{A} : \mathcal{B} \to \mathcal{B}$ is in general not continuous

Proof. First (6) Let $\varepsilon \in (0,t)$, $t > 0$. Then

$$\begin{aligned}\left\|\frac{P_t u - P_{t-\varepsilon}u}{\varepsilon} - P_t\mathbf{A}u\right\| &\leqslant \left\|P_{t-\varepsilon}\frac{P_\varepsilon u - u}{\varepsilon} - P_{t-\varepsilon}\mathbf{A}u\right\| + \left\|P_{t-\varepsilon}\mathbf{A}u - P_t\mathbf{A}u\right\| \\ &\leqslant \left\|\frac{P_\varepsilon u - u}{\varepsilon} - \mathbf{A}u\right\| + \left\|\mathbf{A}u + P_\varepsilon\mathbf{A}u\right\| \\ &\xrightarrow{\varepsilon\downarrow 0} 0 + 0 \text{ b/o } u \in \mathcal{D}(\mathbf{A}) \text{ and (3).}\end{aligned}$$

shows (??) for left-derivative. The right-derivative is easier. That $\mathbf{A}P_t = P_t\mathbf{A}$ follows from

$$\mathbf{A}P_t u = \lim_{s\downarrow 0}\frac{P_sP_t - P_t}{s}u = \lim_{s\downarrow 0}\frac{P_tP_s - P_t}{s}u = P_t\mathbf{A}u \quad \forall u \in \mathfrak{D}(\mathbf{A}).$$

Next (7). Since the continuity of the operator and linearity

$$\begin{aligned}\frac{P_\varepsilon - \mathrm{id}}{\varepsilon}\int_0^t P_s u\,\mathrm{d}s &= \frac{1}{\varepsilon}\int_0^t (P_{s+\varepsilon} - P_s)u\,\mathrm{d}s \\ &= \frac{1}{\varepsilon}\int_t^{t+\varepsilon} P_s u\,\mathrm{d}s - \frac{1}{\varepsilon}\int_0^\varepsilon P_s u\,\mathrm{d}s \qquad \xrightarrow[\text{Lemma ??}]{\varepsilon\downarrow 0} P_t u - u\end{aligned}$$

Since the limit exists, we get $\int_0^t P_s u\,\mathrm{d}s \in \mathfrak{D}(\mathbf{A})$ and the value of the limit is given by: $\mathbf{A}\int_0^t P_s u\,\mathrm{d}s = P_t u - u. \implies$ (7) and 1st «=» of (8) $\forall u \in \mathfrak{B}$.
Now (8): Let $u \in \mathfrak{D}(\mathbf{A})$. Then

$$\begin{aligned}\int_0^t P_s\mathbf{A}u\,\mathrm{d}s &\overset{(6)}{=} \int_0^t \mathbf{A}P_s u\,\mathrm{d}s \\ &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}P_s u\,\mathrm{d}s = P_t u - u.\end{aligned}$$

■

Exercise $u \in C_b^1([a,b],\mathfrak{B}) : \int_a^b u'(s)\mathrm{d}s = u(b) - u(a)$. *Hint:* uniform continuity gives $\int \lim = \lim \int$, use $\|\int \dots\| \leqslant \int \|\cdot\| =$ classical Riemann integral.

Problem with A $\mathfrak{D}(\mathbf{A}) \subsetneq \mathfrak{B}$, $\mathbf{A} : \mathfrak{D}(\mathbf{A}) \subset \mathfrak{B} \to \mathfrak{B}$ is, in general, not bounded. Typical example: $\mathfrak{B} = C_b$, $\mathfrak{D}(\mathbf{A}) = C_b^2$, $\mathbf{A} = \Delta$, u'' cannot be controlled by u.

srg-25 **2.5 Lemma.** $(P_t)_{t\geqslant 0}$ *(C_0)-contraction semigroup, generator $(\mathbf{A}, \mathfrak{D}(\mathbf{A}))$. Then*

a) $\mathfrak{D}(\mathbf{A}) \subset \mathfrak{B}$ dense
b) $(\mathbf{A}, \mathfrak{D}(\mathbf{A}))$ closed, i.e.

$$\left.\begin{array}{r}(u_n)_n \subset \mathfrak{D}(\mathbf{A}) \\ (\mathbf{A}u_n)_n \text{ is Cauchy and} \\ u_n \xrightarrow{\mathfrak{B}} u\end{array}\right\} \quad \begin{array}{c}(1) u \in \mathfrak{D}(\mathbf{A}) \\ (2) \mathbf{A}u = \lim_{n\to\infty}\mathbf{A}u_n\end{array}$$

Compare the lefthand side of (b) with 1.4 f) and the righthand side as continuity of $\mathbf{A}$ gives closedness of $\mathbf{A}$.

Proof. a) Let $u \in \mathfrak{B}$. Set $\varepsilon = \frac{1}{n}$

$$u_\varepsilon = \underbrace{\frac{1}{\varepsilon}\int_0^\varepsilon P_s u\,\mathrm{d}s}_{\in\mathfrak{D}(\mathbf{A})\text{by (7)}} \xrightarrow[\text{Lemma ??}]{\varepsilon\downarrow 0} u$$

b) Let $(u_n)_n \subset \mathfrak{D}(\mathbf{A})$ as in the statement. Then $\exists w \in \mathcal{B} : \mathbf{A}u_n \xrightarrow{n\uparrow\infty} w$ (Cauchy, complete) and

$$
\begin{aligned}
P_t u - u &= \lim_n (P_t u_n - u_n) \\
&\overset{L2.4}{=} \lim_n \int_0^t P_s \mathbf{A} u_n \mathrm{d}s \\
&= \int_0^t P_s w \mathrm{d}s
\end{aligned}
$$

Applying Lemma **??** yields

$$\Longrightarrow \frac{1}{t}(P_t u - u) = \frac{1}{t}\int_0^t P_s w \mathrm{d}s \xrightarrow[Lemma 2.3]{t\downarrow 0} w$$

$\overset{(1)}{\Longrightarrow} u \in \mathfrak{D}(\mathbf{A})$ by existence of the limit, (2) $\mathbf{A}u = w$ by value of the limit.

■

Exercise (P_t), (T_t) are 2 (C_0)-contraction semigroups on $\mathcal{B}$

- $s \mapsto P_s T_{t-s}$ are differentiable, $s < t$
- find the derivative. Keep in mind: $\mathbf{A}, \mathbf{B}$ are generators, so $\mathbf{A} = \mathbf{B} \Longleftrightarrow P_t = T_t$.

srg-26 **2.6 Definition.** $(P_t)_{t\geqslant 0}$ (C_0-)contraction semigroup. Set

$$U_\alpha u := \int_0^\infty e^{-\alpha t} P_t u \mathrm{d}t, \quad u \in \mathcal{B}, \alpha > 0. \tag{9}$$ eq::srg9

Then $(U_\alpha)_{\alpha>0}$ is the resolvent, U_α the resolvent operator at $\alpha > 0$.

The integral is finite, indeeed:

$$\left\|U_\alpha u\right\| \leqslant \int_0^\infty \left\|e^{-\alpha t} P_t u\right\| \mathrm{d}t \leqslant \int_0^\infty e^{-\alpha t} \|u\| \mathrm{d}t = \frac{1}{\alpha}\|u\|.$$

As $t \mapsto e^{-\alpha t} P_t u$ is continuous, (9) defined on $\mathcal{B}$ a family of bounded linear operators.

srg-27 **2.7 Theorem.** *Let $\alpha > 0$, $u \in \mathcal{B}$ and $(U_\alpha)_{\alpha>0}$ as in Definition 2.6.*

a) $\left\|\alpha U_\alpha u\right\| \leqslant \|u\|$ *(«contraction»)*

b) $\left\|\alpha U_\alpha u - u\right\| \xrightarrow{\alpha\uparrow\infty} 0$ *(some times called «strong continuity»)*

c) $(\alpha - \mathbf{A})^{-1}$ *exists (on $\mathcal{B}$) and* $U_\alpha = (\alpha - \mathbf{A})^{-1}$ *is bounded.*

d) *$\alpha, \beta > 0$, the resolvent identity holds:*

$$U_\alpha u - U_\beta u = (\beta - \alpha) U_\alpha U_\beta, \quad \forall u \in \mathcal{B}.$$

Keep in mind: the Semigroup commutes, so does the resolvent, «anti-commutes».

Proof. a) ✓

b) Usual trick

$$\begin{aligned} \left\| \alpha U_\alpha u - u \right\| &= \left\| \int_0^\infty \alpha e^{-\alpha t} (P_t u - u) \mathrm{d}t \right\| \\ &\leqslant \int_0^\infty \alpha e^{-\alpha t} \left\| P_t u - u \right\| \mathrm{d}t \\ &\overset{s=\alpha t}{=} \int_0^\infty e^{-s} \underbrace{\left\| P_{s/\alpha} u - u \right\|}_{\leqslant 2\|u\|} \mathrm{d}s \xrightarrow[\text{(DOM)}]{\alpha \uparrow 0} 0 \end{aligned}$$

c)

$$\begin{aligned} \frac{1}{\varepsilon} \left(P_\varepsilon U_\alpha u - U_\alpha u \right) &= \frac{1}{\varepsilon} \int_0^\infty e^{-\alpha t} \left(P_{t+\varepsilon} u - P_t u \right) \mathrm{d}t \\ &= \frac{1}{\varepsilon} \int_\varepsilon^\infty e^{-\alpha(s-\varepsilon)} P_s u \mathrm{d}s - \frac{1}{\varepsilon} \int_0^\infty e^{-\alpha s} P_s u \mathrm{d}s \\ &= \underbrace{\frac{e^{-\alpha\varepsilon} - 1}{\varepsilon}}_{\substack{t \downarrow 0 \\ \to \alpha}} \underbrace{\int_0^\infty e^{-\alpha s} P_s u \mathrm{d}s}_{= U_\alpha u} - \underbrace{\frac{e^{-\alpha\varepsilon}}{\varepsilon} \int_0^\varepsilon e^{-\alpha s} P_s u \mathrm{d}s,}_{\substack{\varepsilon \downarrow 0 \\ \to e^{\alpha 0} u}} \end{aligned}$$

So (1) $U_\alpha u \in (\mathbf{A})$ since the limit exists. (2) $\mathbf{A} U_\alpha u = \alpha U_\alpha u - u$ as the value of the limit. So we get

$$(\alpha - \mathbf{A}) U_\alpha = \mathrm{id} \iff U_\alpha \text{ is the left-inverse.}$$

Similarly we get for all $u \in \mathfrak{D}(\mathbf{A})$

$$\begin{aligned} \frac{U_\alpha P_\varepsilon u - U_\alpha u}{\varepsilon} &\xrightarrow[U_\alpha \text{ cts}]{\varepsilon \downarrow 0} U_\alpha \mathbf{A} u, \\ \frac{P_\varepsilon U_\alpha u - U_\alpha u}{\varepsilon} &\xrightarrow[U_\alpha \text{ cts}]{\varepsilon \downarrow 0} \alpha U_\alpha u - u \end{aligned}$$

so $U_\alpha(\alpha - \mathbf{A}) = \mathrm{id} \iff U_\alpha$ is the right-inverse, where we used that $P_\varepsilon \int P_t = \int P_\varepsilon P_t = \int P_t P_\varepsilon$.

d) Using Fubini, we get:

$$\begin{aligned} U_\alpha U_\beta u &= \int_0^\infty \int_0^\infty e^{-\alpha s} e^{-\beta t} P_s P_t u \mathrm{d}s \mathrm{d}t \\ &= U_\beta U_\alpha u \end{aligned}$$

Now use c)

$$\begin{aligned} U_\alpha u - U_\beta u &= \left[(\beta - \mathbf{A})U_\beta U_\alpha - (\alpha - \mathbf{A})U_\alpha U_\beta\right] u \\ &= \left[\beta U_\beta U_\alpha - \mathbf{A}U_\beta U_\alpha + \mathbf{A}U_\alpha U_\beta - \alpha U_\alpha U_\beta\right] u \\ &= (\beta - \alpha)U_\beta U_\alpha u. \end{aligned}$$

■

aim 1 prove $P_t u = \underbrace{e^{t\mathbf{A}}}_{\text{give sense}} u$

srg::28 **2.8 Lemma** (Duhamel's formula). *Let (P_t), (T_t) (C_0)-contraction semigroups, same $\mathcal{B}$, generators $(\mathbf{A}, \mathfrak{D}(\mathbf{A}))$, $(\mathbf{B}, \mathfrak{D}(\mathbf{B}))$, respectively. Then*

$$P_t u - T_t u = \int_0^t P_s(\mathbf{A} - \mathbf{B})T_{t-s}u\,\mathrm{d}s, \quad u \in \mathfrak{D}(\mathbf{A}) \cap \mathfrak{D}(\mathbf{B}). \tag{11}$$ srg:eq11

If $P_t T_t = T_t P_t$, then also

$$\left\|P_t u - T_t u\right\| \leqslant t \left\|(\mathbf{A} - \mathbf{B})u\right\|, \quad u \in \mathfrak{D}(\mathbf{A}) \cap \mathfrak{D}(\mathbf{B}). \tag{12}$$ srg:eq11

Proof. Let $\mathfrak{D}(\mathbf{A}) \cap \mathfrak{D}(\mathbf{B})$,

$$\begin{aligned} P_t u - T_t u &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} P_s T_{t-s} u\,\mathrm{d}s \\ &\overset{\text{Ex.}}{=} \int_0^t \left(P_s\mathbf{A}T_{t-s} - P_s\mathbf{B}T_{t-s}\right) u\,\mathrm{d}s. \end{aligned}$$

If $P_t T_t = T_t P_t \underset{\text{DIY}}{\overset{\text{limit}}{\Longrightarrow}} \mathbf{A}T_t = T_t\mathbf{A}$. Since $\left\|T_{t-s}\right\|, \left\|P_t\right\| \leqslant 1$:

$$\begin{aligned} \left\|P_t u - T_t u\right\| &\leqslant \int_0^t \left\|P_s(\mathbf{A} - \mathbf{B})T_{t-s}u\right\| \mathrm{d}s \\ &\leqslant \int_0^t \left\|(\mathbf{A} - \mathbf{B})u\right\| \mathrm{d}s = t \left\|(\mathbf{A} - \mathbf{B})u\right\|. \end{aligned}$$

■

srg-29 **2.9 Lemma** (Dynkin-Reuter). $(\mathbf{A}, \mathfrak{D}(\mathbf{A}))$ *of a (C_0)-contraction semigroup. Assume that* $(\mathbf{B}, \mathfrak{D}(\mathbf{B}))$, *same $\mathcal{B}$, $\mathfrak{D}(\mathbf{B}) \subset \mathcal{B}$ extends $(\mathbf{A}, \mathfrak{D}(\mathbf{A}))$, i.e.*

$$\mathbf{A} \subset \mathbf{B} \overset{def.}{=} \mathfrak{D}(\mathbf{A}) \subset \mathfrak{D}(\mathbf{B}) \text{ and } \mathbf{B}\big|_{\mathfrak{D}(\mathbf{A})} = \mathbf{A}.$$

If

$$\forall u \in \mathfrak{D}(\mathbf{B}) : \underbrace{\mathbf{B}u = u \Longrightarrow u = 0}_{\mathbf{B}-\mathrm{id}\ \textit{injective.}},$$

then $\mathbf{A} = \mathbf{B}$, i.e. $\mathfrak{D}(\mathbf{A}) = \mathfrak{D}(\mathbf{B})$.

Proof. Let $u \in \mathfrak{D}(\mathbf{B})$ and set $g = u - \mathbf{B}u$ and $h := (\mathrm{id} - \mathbf{A})^{-1} g \in \mathfrak{D}(\mathbf{A})$. Then

$$\begin{aligned} h - \mathbf{B}h = h - \mathbf{A}h = (\mathrm{id} - \mathbf{A})U_1 g = g = u - \mathbf{B}u \\ \Longrightarrow \mathbf{B}(h-u) = h - u \Longrightarrow h = u \end{aligned}$$

So $u \in \mathfrak{D}(\mathbf{A})$, so $\mathfrak{D}(\mathbf{A}) = \mathfrak{D}(\mathbf{B})$. ■

srg-210 **2.10 Theorem** (Hille-Yosida ~1948). $(\mathbf{A}, \mathfrak{D}(\mathbf{A}))$ *linearer Operator generates a* (C_0)*-contraction semigroup iff*

(a) $\mathbf{A}$ *closed*

(b) $\mathfrak{D}(\mathbf{A}) \subset \mathfrak{B}$ *dense*

(c) $(\alpha - \mathbf{A})$ *has bounded inverse* $\forall \alpha > 0$

(d) $\left\|\alpha(\alpha - \mathbf{A})^{-1}u\right\| \leqslant \|u\|$

Proof. $\Longrightarrow$ Clear, see 2.5 and 2.7.

$\Longleftarrow$ Assume a) - d) holds. Set $U_\alpha = (\alpha - \mathbf{A})^{-1}$ and define the **Yosida-Approximation**[2]. For $u \in \mathfrak{B}$:

$$\begin{aligned} \mathbf{A}^{(\alpha)}u &= \alpha(\alpha U_\alpha - 1)u \\ &= \alpha\left(\alpha U_\alpha - (\alpha - \mathbf{A})U_\alpha\right)u \\ &= \alpha \mathbf{A} U_\alpha u \\ &= \alpha U_\alpha(\mathbf{A}u) \xrightarrow[??]{\alpha \uparrow \infty} \mathbf{A}u \quad u \in \mathfrak{D}(\mathbf{A}). \end{aligned} \tag{12}$$

The Yosida-Approximation is bounded:

$$\left\|\mathbf{A}^{(\alpha)}\right\| = \left\|\alpha(\alpha U_\alpha - 1)u\right\| \leqslant \alpha\left(\left\|\alpha U_\alpha u\right\| + \|u\|\right) \leqslant 2\alpha\,\|u\|\,.$$

Can use $\mathbf{A}^{(\alpha)}$ to build a semigroup!

$$\begin{aligned} T_t^{(\alpha)}u := e^{t\mathbf{A}^{(\alpha)}}u &= e^{t\alpha\,\alpha U_\alpha} u\, e^{-t\alpha} \\ &= e^{-t\alpha}\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!}(\alpha U_\alpha)^n u \end{aligned}$$

and we have

$$\left\|T_t^{(\alpha)}u\right\| \leqslant e^{-t\alpha}\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!}\underbrace{\left\|(\alpha U_\alpha)^n u\right\|}_{\|u\|\cdot\left\|\alpha U_\alpha\right\|^n \leqslant \|u\|}$$

This shows:

[2] $\alpha\left(\frac{\alpha}{\alpha - \mathbf{A}} - 1\right) = \alpha\frac{\alpha - \alpha + \mathbf{A}}{\alpha - \mathbf{A}} = \frac{\alpha}{\alpha - \mathbf{A}}\mathbf{A} \xrightarrow{\alpha\uparrow\infty} \mathbf{A}$.

- $\sum_0^\infty$ defined
- $\left(T_t^{(\alpha)}\right)_{t\geqslant 0}$ is semigroup (exp series!), contraction
- $\sum_0^\infty$ converges locally uniform in $t \Longrightarrow$ continuity in t

$\Longrightarrow \left(T_t^{(\alpha)}\right)_{t\geqslant 0}$ is a (C_0)-contraction semigroup.

Play with α U_α, U_β commute, we see

$$T_t^{(\alpha)} T_t^{(\beta)} = T_t^{(\beta)} T_t^{(\alpha)} \text{ and } \underbrace{T_t^{(\alpha)} \text{ has generator } \mathbf{A}^{(\alpha)}}_{\text{Ex. = 1st order term in exp-series}}$$

Using Duhamel's formula we get[3]

$$\left\| T_t^{(\alpha)} u - T_t^{(\beta)} u \right\| \leqslant t \left\| \mathbf{A}^{(\alpha)} u - \mathbf{A}^{(\beta)} u \right\| \xrightarrow{\alpha,\beta\uparrow\infty} 0 \text{ locally uniformly in } t \quad \forall u \in \mathfrak{D}(\mathbf{A})$$

$\Longrightarrow (T_t^{(\alpha)} u)_{\alpha>0}$ Cauchy if $u \in \mathfrak{D}(\mathbf{A}) \subset \mathfrak{B}$

$\Longrightarrow P_t u := \lim_{\alpha\uparrow\infty} T_t^{(\alpha)} u$, $u \in \mathfrak{D}(\mathbf{A})$ exists locally uniformly in t, i.e. P_t inherits from $T_t^{(\alpha)}$. So

- semigroup properties
- contraction
- (C_0)

Problem $\mathfrak{D}(\mathbf{A})$ too small!

Final step extend P_t from $\mathfrak{D}(\mathbf{A})$ to $\mathfrak{B}$ ok as $\mathfrak{D}(\mathbf{A})$ dense and $\left\| P_t u \right\| \leqslant \|u\|$ is (Lipschitz-)cts $\Longrightarrow$ gives semigroup on $(\mathbf{B}, \mathfrak{D}(\mathbf{B}))$[4]

Still open Is $(\mathbf{A}, \mathfrak{D}(\mathbf{A}))$ generator of P_t?

As (P_t) is a (C_0)-contraction semigroup, there exists a generator $(\mathbf{B}, \mathfrak{D}(\mathbf{B}))$. Aim

[3] $\mathbf{A}^{(\alpha)} u \overset{u\mathfrak{D}(\mathbf{A})}{=} \alpha U_\alpha \mathbf{A} u \xrightarrow{\alpha\uparrow\infty} \mathbf{A}u$, and $\mathbf{A}^{(\alpha)} u - \mathbf{A}^{(\beta)} u \to \mathbf{A}u - \mathbf{A}u$ if $u \in \mathfrak{D}(\mathbf{A})$.
[4] Excerise: Check (C_0) on $\mathfrak{B}$ carefully!

is $\mathbf{A} = \mathbf{B}$.

$$\begin{array}{ccc}
\frac{1}{t}\left(T_t^{(\alpha)}u - u\right) & = & \frac{1}{t}\int_0^t T_s^{(\alpha)}\mathbf{A}^{(\alpha)}u\,\mathrm{d}s \\
\Big\downarrow{\scriptstyle \alpha\to\infty} & & \Big\downarrow{\scriptstyle \alpha\to\infty} \\
\frac{1}{t}\left(P_t u - u\right) & = & \frac{1}{t}\int_0^t P_s\mathbf{A}u\,\mathrm{d}s \\
\Big\downarrow{\scriptstyle t\downarrow 0} & & \Big\downarrow{\scriptstyle \lim\exists} \\
\frac{\mathrm{d}}{\mathrm{d}t}P_t u\Big|_{t=0} & = & \mathbf{A}u
\end{array}$$

$\Longrightarrow \mathbf{B} \supset \mathbf{A}$. Since $(\mathrm{id} - \mathbf{B})$ is invertible (Reason for $(P_t)_t$ a) - d) are necessary for the (C_0)-contraction semigroup with generator $\mathbf{B}$, we can use Lemma 2.9 to get $\mathbf{A} \subset \mathbf{B} \Longrightarrow \mathbf{B} = \mathbf{A}$. ■

srg-211 **2.11 Remark.** Duhamel shows $\Longrightarrow$ of

$$\mathbf{A} = \mathbf{B},\ \mathbf{A}, \mathbf{B} \text{ generate } P_t \text{ and } T_t \Longleftrightarrow P_t = T_t,$$

and $\Longleftarrow$ is clear by definition of $\mathbf{A}$ and $\mathbf{B}$.

srg-212 **2.12 Lemma.** *Let $(\mathbf{G}_\alpha)_{\alpha>0}$ on $\mathcal{B}$ be a family of operator so that*

$$\alpha\mathbf{G}_\alpha u \xrightarrow{\alpha\uparrow\infty} u, \quad u \in \mathcal{B} \tag{13}$$ srg::eq13

$$\left\|\alpha\mathbf{G}_\alpha u\right\| \leqslant \|u\|, \quad u \in \mathcal{B}, \alpha > 0 \tag{14}$$ srg::eq14

$$\mathbf{G}_\alpha u - \mathbf{G}_\beta u = (\beta - \alpha)\mathbf{G}_\alpha\mathbf{G}_\beta u, \quad u \in \mathcal{B} \tag{15}$$ srg::eq15

(«Pseudo-resolvent»). Then $\exists!\,(\mathbf{L}, \mathcal{D}(\mathbf{L}))$ so that $(\alpha - \mathbf{L})$ is invertible and $\mathbf{G}_\alpha = (\alpha - \mathbf{L})^{-1}$, $\mathbf{L}$ closed, $\mathcal{D}(\mathbf{L})$ dense.

Remark This finished the programme $P_t \overset{1:1}{\longleftrightarrow} \mathbf{A} \overset{1:1}{\longleftrightarrow} U_\alpha$ (by Hille-Yosida and Lemma 2.12)

Proof. Set $\mathcal{D}(\mathbf{L}) = \mathbf{G}_\alpha(\mathbf{B}) = \left\{\mathbf{G}_\alpha u : u \in \mathcal{B}\right\}$, b/o (15): $\mathbf{G}_\alpha(\mathcal{B}) = \mathbf{G}_\beta(\mathcal{B})\ \forall\alpha, \beta > 0$, so $\mathcal{D}(\mathbf{L})$ independent of α.
$\mathbf{G}_\alpha$ injective α is fixed, $\mathbf{G}_\alpha u = 0 \Longrightarrow u = 0$.
Idea (15) then

$$\begin{aligned}
\mathbf{G}_\alpha u = 0 \text{ some } \alpha &\Longrightarrow \mathbf{G}_\beta u = 0 \quad \forall\beta > 0 \\
&\Longrightarrow u \overset{??}{\leftarrow} \beta\mathbf{G}_\beta u = 0 \Longrightarrow u = 0
\end{aligned}$$

So $\mathbf{G}_\alpha : \mathcal{B} \to \mathcal{D}(\mathbf{L})$ bijective, invertible. Define $\mathbf{L}$ on $\mathcal{D}(\mathbf{L})$:

$$\begin{aligned} \mathbf{L}u &= \alpha u - f \quad \text{if } u = \mathbf{G}_\alpha f \in \mathcal{D}(\mathbf{L}),\ f \in \mathcal{B}, \\ &= (\alpha - \mathbf{G}_\alpha^{-1})u \end{aligned}$$

Is it well-defined, i.e. independent of α? Show

$$\alpha - \mathbf{G}_\alpha^{-1} = \beta - \mathbf{G}_\beta^{-1} \quad (\alpha, \beta > 0 \text{ on } \mathcal{D}(\mathbf{L})).$$

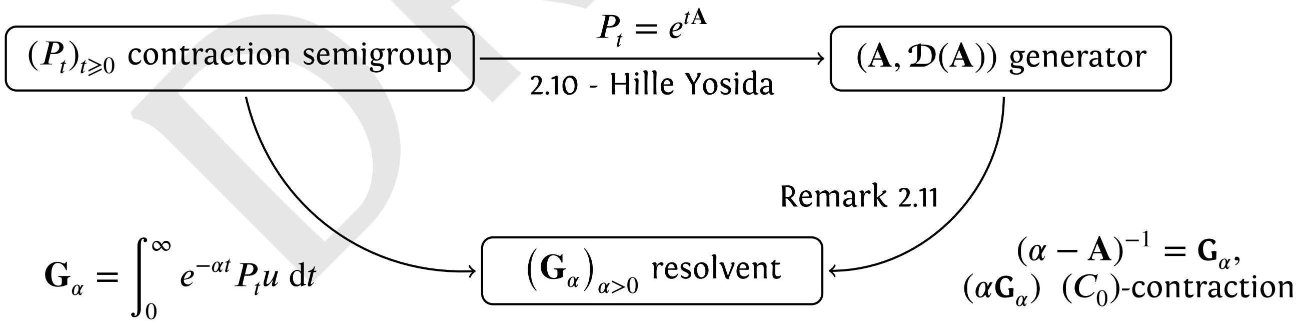
Take $u \in \mathbf{G}_\alpha(\mathcal{B})\ \exists f \in \mathcal{B} : u = \mathbf{G}_\alpha f$ and

$$\begin{aligned} &\mathbf{G}_\beta \left((\alpha - \mathbf{G}_\alpha^{-1})u - (\beta - \mathbf{G}_\beta^{-1})u\right) \\ &\qquad = \alpha \mathbf{G}_\beta \mathbf{G}_\alpha f - \mathbf{G}_\beta f - \beta \mathbf{G}_\beta \mathbf{G}_\alpha f + \mathbf{G}_\alpha f \\ &\qquad \overset{(15)}{=} 0, \end{aligned}$$

Since $\mathbf{G}_\beta$ is injective, $\alpha - \mathbf{G}_\alpha^{-1} = \beta - \mathbf{G}_\beta^{-1}$. ■

Remark $\mathcal{D}(\mathbf{L})$ dense b/o (13) and $(\mathbf{L}, \mathcal{D}(\mathbf{L}))$ closed (clear, inverse of a bounded operator).

Direct proof $\mathcal{D}(\mathbf{L}) \ni u_n \to u$ and $\mathbf{L}u_n \to w \implies (\alpha - \mathbf{L})u_n \to \alpha u - u$ and show $u \in \mathcal{D}(\mathbf{L})$, $Lu = w$. But $u = \lim_n u_n = \lim_n \mathbf{G}_\alpha(\alpha - \mathbf{L})u_n = \mathbf{G}_\alpha(\alpha u - w) \implies u \in \mathcal{D}(\mathbf{L})$ and $(\alpha - \mathbf{L})u = \alpha u - w \implies Lu = w$.



Now $(\mathbf{A}) \subsetneq \mathcal{F}$ and $u, v \in \mathcal{F} : \mathcal{E}(u, v) = \langle -\mathbf{A}u, v \rangle$

Chapter 3

Semigroups and Forms

closed form $(\mathcal{E},\mathcal{F}) := \mathcal{F} \subset L^2(X,m)$, bilinear, lower bdd $(\mathcal{E}1)$, bound $\gamma \geqslant 0$, sectorial $(\mathcal{E}2)$, closed $(\mathcal{E}3)$. Need Stampacchia's theorem **??**

$$\exists!\ v \in \Gamma \forall w \in \Gamma : \mathcal{E}_\alpha(v, w-v) \geqslant \mathcal{J}(w-v) \quad (1) \quad \boxed{\texttt{sf::eq1}}$$

$$\text{If } \Gamma = \mathcal{F} = \text{vector space, then} \quad (3.1)$$

$$\exists!\ v \in \mathcal{F} \forall w \in \mathcal{F} : \mathcal{E}_\alpha(v, w) \geqslant \mathcal{J}(w) \quad (1') \quad \boxed{\texttt{sf::eq1-}}$$

So for vector spaces (1) $\iff$ (1′), comes from $\mathcal{F} = -\mathcal{F}$; and (1′) = Lax-Milgram theorem

Idea $\quad \mathcal{E}_\alpha(v,w) = \langle(\alpha - \mathbf{A})v, w\rangle_{L^2} = \langle f, w\rangle_{L^2} = \mathcal{J}(w)$
$\iff \forall w \in \mathcal{F} : (\alpha - \mathbf{A})v = f$
$\overset{\text{Lax-Milgram}}{\Longrightarrow}$ $\exists$ solution to this *weakly* $v \in \mathcal{F}$, not $v \in \mathcal{D}(\mathbf{A})$.

We will check «(c)» in Hille-Yosida 2.10, i.e. «$(\alpha - \mathbf{A})$ has a bdd inverse», i.e.
$\iff (\alpha - \mathbf{A})u = f$ and find u for («many») f[1]

sf-21 **3.1 Theorem.** *$(\mathcal{E},\mathcal{F})$ closed form on $L^2(X,m)$. Then $\exists a$ (C_0)-contraction semigroups $(T_t)_t$, $(\hat{T}_t)_t$ on $L^2(X,m)$, satisfying*

- $\|T_t f\|_{L^2} \leqslant e^{t\gamma}\|f\|_{L^2}$, $\quad \|\hat{T}_t f\|_{L^2} \leqslant e^{t\gamma}\|f\|_{L^2}$
- *duality* $\langle T_t f, g\rangle_{L^2} = \left\langle f, \hat{T}_f g\right\rangle_{L^2}$

Further for $\alpha > \gamma$ exists the resolvents

- $G_\alpha f = \int_0^\infty e^{-\alpha t} T_f f \,\mathrm{d}t$, $\quad \hat{G}_\alpha f = \int_0^\infty e^{-\alpha t}\hat{T}_f f\,\mathrm{d}t$

and

$$\mathcal{E}_\alpha(G_\alpha f, u) = \langle f, u\rangle_{L^2} = \mathcal{E}_\alpha(u, \hat{G}_\alpha f) \quad \forall f \in L^2(m), u \in \mathcal{F} \subset L^2(m), \alpha > \gamma \quad (2) \quad \boxed{\texttt{sf::eq2}}$$

An attempt for general notation $\quad f, g, h \in L^2(m)$ and $u, v, w \in \mathcal{F}$ or $\mathcal{D}(\mathbf{A})$

Proof. Strategy $\mathcal{E} \xrightarrow{(1')} \mathbf{G}_\alpha \xrightarrow{2.12} \mathbf{A} \xrightarrow{2.10} T_t \longrightarrow \mathbf{G}_\alpha$

[1] Keep in mind we mostly get only dense.

1° Use (1′) with $\mathcal{J}(\cdot) = \langle f, \cdot \rangle_{L^2(m)}$ ($\mathcal{E}_\alpha$-cts as $\alpha > \gamma$)

$$\exists!\, v = \underbrace{\mathbf{G}_\alpha(f)}_{\text{any fn of } \alpha, f} \in \mathcal{F}\ \forall u \in \mathcal{F} : \mathcal{E}_\alpha(\mathbf{G}_\alpha f, u) = \langle f, u \rangle_{L^2} \quad (3)$$ sf::eq3

$$\exists!\, \hat{v} = \hat{\mathbf{G}}_\alpha(f) \in \mathcal{F} \forall \hat{u} \in \mathcal{F} : \mathcal{E}_\alpha(\hat{u}, \hat{\mathbf{G}}_\alpha g) = \langle \hat{u}, g \rangle_{L^2} \quad (4)$$ sf::eq4

(3.2)

Claim $f \mapsto \mathbf{G}_\alpha f$ is linear. Consequence of uniqueness in (3) and linearity of $\mathcal{E}_\alpha(\cdot, u)$, $\langle \cdot, u \rangle_{L^2}$

datum f	solution v
f	$\mathbf{G}_\alpha f$
φ	$\mathbf{G}_\alpha \varphi$
$f + \varphi$	$\mathbf{G}_\alpha f + \mathbf{G}_\alpha \varphi$
$f + \varphi$	$\mathbf{G}_\alpha(f + \varphi)$

so the third line gives (uniqueness!) $\mathbf{G}_\alpha f + \mathbf{G}_\alpha \varphi = \mathbf{G}_\alpha(f + \varphi)$ + additivity, rest (homogenety) = exercise!

2° $\langle \mathbf{G}_\alpha f, f \rangle_{L^2} \overset{(4)}{=} \mathcal{E}_\alpha(\mathbf{G}_\alpha f, \hat{\mathbf{G}}_\alpha g) \overset{(3)}{=} \left\langle f, \hat{\mathbf{G}}_\alpha g \right\rangle_{L^2}$

3° Resolvent equation for $(\mathbf{G}_\alpha)_{\alpha > \gamma}$. Pick $\alpha, \beta > \gamma\ \forall v \in \mathcal{F}$. Then $\forall f \in L$:

$$\begin{aligned}
&\mathcal{E}_\alpha\left(\underline{\mathbf{G}_\beta f - (\alpha - \beta)\mathbf{G}_\alpha \mathbf{G}_\beta f}, v\right) \\
&\quad = \mathcal{E}_\beta(\mathbf{G}_\beta f, v) + (\alpha - \beta)\left\langle \mathbf{G}_\beta f, v \right\rangle_{L^2} - (\alpha - \beta)\mathcal{E}_\alpha(\mathbf{G}_\alpha \mathbf{G}_\beta f, v) \\
&\quad = \langle f, v \rangle_{L^2} + (\alpha - \beta)\left\langle \mathbf{G}_\beta f, v \right\rangle_{L^2} - (\alpha - \beta)\left\langle \mathbf{G}_\beta f, v \right\rangle_{L^2} \\
&\quad = \langle f, v \rangle_{L^2} \\
&\quad = \mathcal{E}_\alpha(\underline{\mathbf{G}_\alpha f}, v)
\end{aligned}$$

and the underlined stuff is equal as $\mathcal{E}_\alpha(\cdot, \cdot)$ is a scalar product![2]

4° **Contractivity** $\alpha = \gamma + \varepsilon$, $\varepsilon > 0$. Then

$$\begin{aligned}
\varepsilon \left\| \mathbf{G}_{\gamma+\varepsilon} f \right\|_{L^2}^2 &\leqslant \mathcal{E}_{\gamma+\varepsilon}(\mathbf{G}_{\gamma+\varepsilon} f, \mathbf{G}_{\gamma+\varepsilon} f) \\
&= \left\langle f, \mathbf{G}_{\gamma+\varepsilon} f \right\rangle_{L^2} \\
&\overset{\text{CSI}}{\leqslant} \| f \|_{L^2} \left\| \mathbf{G}_{\gamma+\varepsilon} f \right\|_{L^2},
\end{aligned}$$

now divide $\left\| \varepsilon \mathbf{G}_{\gamma+\varepsilon} f \right\|_{L^2} \leqslant \| f \|_{L^2}$, i.e. $(\varepsilon \mathbf{G}_{\gamma+\varepsilon})_{\varepsilon > 0}$[3].

5° All steps from above apply to $\hat{\mathbf{G}}_\alpha$, too. $\implies \mathcal{E}_\alpha(\mathbf{G}_\alpha f, u) = \langle f, u \rangle_{L^2} = \langle u, f \rangle_{L^2} = \mathcal{E}_\alpha(u, \hat{\mathbf{G}}_\alpha f)$

[2] Excerise: $\mathcal{H}$ Hilbert space, $(\cdot, \cdot)_{\mathcal{H}}$. Then $(\mathbf{A}, h)_{\mathcal{H}} = (\mathbf{B}, h)_{\mathcal{H}}\ \forall h \implies \mathbf{A} = \mathbf{B}$

[3] Keep in mind $\iff \left\| \mathbf{G}_{\gamma+\varepsilon} f \right\|_{L^2} \leqslant \frac{1}{\varepsilon} \| f \|_{L^2} \iff \left\| \mathbf{G}_\alpha f \right\|_{L^2} \leqslant \frac{1}{\alpha - \gamma} \| f \|_{L^2}$

6° **Generator** $(\mathbf{A}^{(\gamma)}, \mathfrak{D}(\mathbf{A}^{(\gamma)})$ as in 2.12. Use family $(\mathbf{G}_{\gamma+\varepsilon})_{\varepsilon>0}$. Need (2.13)-(2.15) stated in 2.12.

- $\mathfrak{D}(\mathbf{A})$ dense, where $\mathbf{G}_\alpha(L^2) := \mathfrak{D}(\mathbf{A})$, $\alpha > \gamma$. LHS indenpendent of α b/o resolvent equation.
  Let $g \in L^2(m)$. Assume $\forall f \in L^2(m) : \langle \mathbf{G}_\alpha f, g\rangle = 0$.[4] Then:

$$\begin{aligned} &\Longrightarrow \forall f \in L^2(m) : && \left\langle f, \hat{\mathbf{G}}_\alpha g \right\rangle_{L^2} = 0 \\ &\Longrightarrow && \hat{\mathbf{G}}_\alpha g = 0 \\ &\Longrightarrow \forall u \in \mathcal{F} : && \langle u, g\rangle_{L^2} = \mathcal{E}_\alpha(u, \hat{\mathbf{G}}_\alpha g) = 0 \\ &\Longrightarrow && g = 0, \end{aligned}$$

  so $\mathbf{G}_\alpha(L^2)$ dense in $L^2(m)$. Now check (2.13) «strong continuity»: $\alpha, \beta > \gamma$:

$$\begin{aligned} \left\|\alpha \mathbf{G}_\alpha \mathbf{G}_\beta f - \mathbf{G}_\beta f\right\|_{L^2} &\overset{\text{res. eqn}}{=} \left\|\mathbf{G}_\alpha f - \beta \mathbf{G}_\alpha \mathbf{G}_\beta f\right\|_{L^2} \\ &= \left\|\mathbf{G}_\alpha (f - \beta \mathbf{G}_\beta f)\right\|_{L^2} \\ &\overset{4^\circ}{\leqslant} \frac{1}{\alpha - \gamma} \left\|(f - \beta \mathbf{G}_\beta f)\right\|_{L^2} \xrightarrow[\beta \text{ fixed}]{\alpha,\beta\uparrow\infty} 0 \end{aligned}$$

  $\Longrightarrow \alpha \mathbf{G}_\alpha(\mathbf{G}_\beta f) \xrightarrow{\alpha\uparrow\infty} \mathbf{G}_\beta f$, $\alpha \mathbf{G}_\alpha u \xrightarrow{\alpha\uparrow\infty} u \ \forall u \in \mathbf{G}_\beta(L^2) = \overline{\mathfrak{D}(\mathbf{A})}$
  $\Longrightarrow \alpha \mathbf{G}_\alpha f \to f \ \forall f \in L^2 = \mathfrak{D}(\mathbf{A})$.
- Now use 2.12 to define $(\mathbf{A}^{(\gamma)}, \mathfrak{D}(\mathbf{A}^{(\gamma)}))$: $\mathbf{A}^{(\gamma)} := \mathcal{E} - \mathbf{G}_{\gamma+\varepsilon}^{-1}$ and know $(\mathbf{A}^{(\gamma)}, \mathfrak{D}(\mathbf{A}))$ densely defined, closed operator $\mathbf{G}_{\gamma+\varepsilon} = (\mathcal{E} - \mathbf{A}^{(\gamma)})^{-1}$.

7° Now use 2.10 (Hille-Yosida) to get a (C_0)-contraction semigroup $(T_t^{(\gamma)})_{t\geqslant 0}$ generated by $(\mathbf{A}^{(\gamma)}, \mathfrak{D}(\mathbf{A}))$. Resolvent $(\mathbf{G}_{\gamma+\varepsilon})_{\varepsilon>0}$.

Getting rid of γ $\quad T_t := e^{\gamma t} T_t^{(\gamma)}$ (with the trivial semigroup, generator γ)

T_t is again a semigroup (direct proof) and (C_0) b/o $t \mapsto e^{\gamma t}$ is cts.

$$\left\|T_t\right\| = \sup_{f\neq 0} \frac{\langle T_t, f\rangle_{L^2}}{\|f\|_{L^2}} = \left\|e^{\gamma t} T_t^{(\gamma)}\right\| = e^{\gamma t} \left\|T_t^{(\gamma)}\right\| = e^{\gamma t}.$$

Then for $\lambda > \gamma$

$$\begin{aligned} U_\lambda f &= \int_0^\infty e^{-\lambda t} e^{\gamma t} T_t^{(\gamma)} f \,\mathrm{d}t \\ &= \int_0^\infty e^{-(\lambda-\gamma)t} T_t^{(\gamma)} f \,\mathrm{d}t \\ &= \mathbf{G}_{\gamma+(\lambda-\gamma)} f \\ &\Longrightarrow U_\lambda f = \mathbf{G}_\lambda f \quad \forall \lambda > \gamma \end{aligned}$$

[4] Need $g = 0$. Excerise: $(\mathcal{H}, (\cdot,\cdot))$ Hilbert space, then $(\forall u \in \mathfrak{D} \subset \mathcal{H} : (u,g)_{\mathcal{H}} = 0 \Longrightarrow g = 0) \Longrightarrow \mathfrak{D}$ dense.

And finally, the generator of T_t: Differentiate at $t = 0 : e^{\gamma t} T_t^{(\gamma)}$

$$\mathbf{A} = \mathbf{A}^{(\gamma)} + \gamma$$

8° The previous steps apply to $\hat{T}_t$, $\hat{\mathbf{G}}_\alpha$, $\hat{L}$ literally.

■

Note

(1) $\mathcal{E}$ generates a semigroup

(2) $\gamma \leqslant 0 \Longleftrightarrow$ semigroup contractive

(3) **Resolvent set** can be shifted: $\varrho(\mathbf{B}) = \{z \in \mathbb{C} : (z - \mathbf{B}) \text{ has a bounded inverse}\}$. We have seen $(\gamma, +\infty) \subset \varrho(\mathbf{A}) : \mathbf{A} \longleftrightarrow \mathcal{E}$

ultimate aim study stochastic processes $X_t(\omega)$

- naive X_t is a random object evolving in time t («movement»)
- P_t semigroup is also «evolution»

$$P_{t+s} = P_t P_s$$

  so naive start in 0 und move to $t + s$. So the semigroup property says start in 0 over P_s and stop in s as some initial condition (so the PDE does not change character, need unique solution) then move with P_t to $t + s$.

- stochastic process: «Markov property»

$$«X_{t+s} = \Phi(X_s, X_t \circ \vartheta_s)»$$

  The idea with semigroups. Assume P_t is given by

$$\begin{aligned} P_t f(x) &= \int f(y) p_t(x, \mathrm{d}y) \text{ integral operator} \\ P_t P_s f(x) &= \int P_s f(y) p_t(x, \mathrm{d}y) = \int\int f(z) p_s(y, \mathrm{d}z) p_t(x, \mathrm{d}y) \\ P_{t+s} f(x) &= \int\int f(z) p_{t+s}(x, \mathrm{d}z) \end{aligned}$$

so we get the Chapman-Kolmogorv equation

$$p_{t+s}(x, \mathrm{d}z) = \int p_s(y, \mathrm{d}z) p_t(x, \mathrm{d}y)$$

$$p_{t+s}(x, B) = \int_X \underbrace{p_s(y, B)}_{\overset{!}{\leqslant} 1} p_t(x, \mathrm{d}y), \quad \text{where } B \subset \mathfrak{X} \text{ Borel}$$

Idea

$$\begin{aligned} p_{t+s}(x, B) &= \text{ probability to begin at } t = 0 \text{ in } x \text{ and to move in } (t+s) \text{ time in } B \\ &= \mathbb{P}^x(X_{t+s} \in B) \leqslant 1, \end{aligned}$$

but the last equation means Chapman-Kolmogorov does not create mass!

Assume $p_t(x, \cdot)$ is a measure so that the mass $\leqslant 1$ «sub-probability»

Consequence $P_t f(x) \geqslant 0$, if $f \geqslant 0$ and $P_t f(x) \leqslant 1$ if $f \leqslant 1$. Both together we call (sub)Markov property of $(P_t)_{t \geqslant 0}$

Now all semigroups are sub-Markov

DIY If $(T_t)_{t\geqslant 0}$ is sub-Markovian, then $(\alpha \mathbf{G}_\alpha)_{\alpha>0}$ is sub-Markovian. Converse also holds.

sf-32 **3.2 Definition.** Let $(\mathcal{E}, \mathcal{F})$ be a closed form with resolvent $(\mathbf{G}_\alpha)_{\alpha>0}$. The **approximate form** is

$$\mathcal{E}^\alpha(f, g) := \alpha \ \langle f - \alpha \mathbf{G}_\alpha f, g\rangle_{L^2} \quad \big(f, g \in L^2(m)\big)\,. \tag{3.3}$$ sf::eq05

sf-33 **3.3 Remark.** (i) **Notation** $\mathcal{E}^\alpha_\lambda(\cdot,\cdot) := \mathcal{E}^\alpha(\cdot,\cdot) + \lambda \langle \cdot, \cdot \rangle_{L^2}$

(ii) **Useful identity**

$$\begin{aligned} \mathcal{E}^\alpha(f, v) &= \alpha \langle f - \alpha \mathbf{G}_\alpha f, v\rangle_{L^2} = \alpha \langle f, v\rangle_{L^2} - \alpha \langle \alpha \mathbf{G}_\alpha f, v\rangle_{L^2} \\ &= \alpha \mathcal{E}_\alpha(\mathbf{G}_\alpha f, v) - \alpha \langle \alpha G_\alpha, v\rangle_{L^2} \\ &= \mathcal{E}(\alpha \mathbf{G}_\alpha f, v) \end{aligned} \tag{3.4}$$ sf::eq06

(iii) **Philosophy** $\mathcal{E}^\alpha =$ «Yosida approximation». Since, formally,

$$\alpha(1 - \alpha \mathbf{G}_\alpha) = \alpha \left(1 - \frac{\alpha}{\alpha - L}\right) = -\frac{\alpha L}{\alpha - L} = -\alpha \ L \ \mathbf{G}_\alpha$$

we expect

$$\langle -\alpha\ L\ \mathbf{G}_\alpha, g\rangle_{L^2} \xrightarrow{\alpha\uparrow\infty} \langle -Lf, g\rangle\,.$$

Ok, if $f \in \mathfrak{D}(L)$, $g \in L^2(m)$.

sf-34 **3.4 Lemma.** *Let $\alpha > \gamma$, $\lambda \geqslant \gamma\left(\frac{\alpha}{\alpha-\gamma}\right)^2$, $f, g \in L^2(m)$, $u, v \in \mathfrak{F}$. Then*

$$\mathcal{E}(\alpha\mathbf{G}_\alpha f, \alpha\mathbf{G}_\alpha f) \leqslant \mathcal{E}^\alpha(f, f), \tag{3.5}$$ sf::eq37

$$\mathcal{E}(\alpha\hat{\mathbf{G}}_\alpha f, \alpha\hat{\mathbf{G}}_\alpha f) \leqslant \mathcal{E}^\alpha(f, f), \tag{3.$\hat{8}$}$$ sf::eq37

$$|\mathcal{E}^\alpha(f, v)| \leqslant \kappa\sqrt{\mathcal{E}^\alpha_\lambda(f, f)}\sqrt{\mathcal{E}_\gamma(v, v)}, \tag{3.6}$$ sf::eq38

$$|\mathcal{E}^\alpha(u, g)| \leqslant \kappa\sqrt{\mathcal{E}_\gamma(u, u)}\sqrt{\mathcal{E}^\alpha_\lambda(g, g)}, \tag{3.$\hat{8}$}$$ sf::eq38

$$|\mathcal{E}^\alpha(u, u)| \leqslant \kappa^2\mathcal{E}_\gamma(u, u) + \kappa\sqrt{\lambda}\ \|u\|_{L^2}\sqrt{\mathcal{E}_\gamma(u, u)}. \tag{3.7}$$ sf::eq39

$$\mathcal{E}(\alpha\mathbf{G}_\alpha u) \leqslant \mathcal{E}_\alpha(u)) \overset{(3.7)}{\leqslant} \kappa^2\mathcal{E}(u) \tag{3.8}$$ sf::eq31

If $\gamma = 0$, we can take $\lambda \downarrow 0$ and (3.7) gives

$$\mathcal{E}(\alpha\mathbf{G}_\alpha u, \alpha\mathbf{G}_\alpha u) \leqslant \mathcal{E}^\alpha(u, u) \overset{(3.7)}{\leqslant} \kappa^2\mathcal{E}(u, u).$$

Proof. 3.5 : By (**??**),

$$\begin{aligned}\mathcal{E}(\alpha\mathbf{G}_\alpha f, \alpha\mathbf{G}_\alpha f) = \mathcal{E}^\alpha(f, \mathbf{G}_\alpha f) &\overset{\text{def}}{=} \alpha\,\langle f - \alpha\mathbf{G}_\alpha f, \alpha\mathbf{G}_\alpha f \pm f\rangle_{L^2}\\ &= -\alpha\left\|f - \alpha\mathbf{G}_\alpha f\right\|^2_{L^2} + \alpha\,\langle f - \alpha\mathbf{G}_\alpha f, f\rangle_{L^2} \leqslant \mathcal{E}^\alpha(f, f).\end{aligned}$$

3.$\hat{8}$ Similar.

3.6 By (**??**) and ($\mathcal{E}_2$),

$$|\mathcal{E}^\alpha(f, v)| \overset{??}{=} \left|\mathcal{E}(\alpha\mathbf{G}_\alpha f, v)\right| \overset{(\mathcal{E}_2)}{=} \kappa\sqrt{\mathcal{E}_\gamma(\alpha\mathbf{G}_\alpha f)}\sqrt{\mathcal{E}_\gamma(v)}.$$

Since,

$$\begin{aligned}\mathcal{E}_\gamma(\alpha\mathbf{G}_\alpha f, \alpha\mathbf{G}_\alpha f) &\overset{\text{def}}{=} \mathcal{E}(\alpha\mathbf{G}_\alpha f, \alpha\mathbf{G}_\alpha f) + \gamma\left\|\alpha\mathbf{G}_\alpha f\right\|^2_{L^2}\\ &\overset{3.5}{\leqslant} \mathcal{E}^\alpha(f, f) + \gamma\left(\frac{\alpha}{\alpha-\gamma}\right)^2\|f\|_{L^2}\\ &\leqslant \mathcal{E}^\alpha_\lambda(f, f)\end{aligned}$$

the claim follows. For the second inequality, we used

$$\left\|\alpha\mathbf{G}_\alpha f\right\|_{L^2} \leqslant \frac{\alpha}{\alpha-\gamma}\|f\|_{L^2}\,,$$

see step four in the proof of Theorem **??**.

3.7 Use $f = v = u$ in (3.6):

$$|\mathcal{E}^\alpha(u,u)|^2 \leqslant \kappa^2 \left(\mathcal{E}^\alpha(u,u) + \lambda \|u\|_{L^2}^2\right) \mathcal{E}_\gamma(u,u).$$

This is an inequality of the form

$$x^2 \leqslant (x+a)2b, \quad a := \lambda \|u\|_{L^2}^2, \; 2b := \kappa^2 \, \mathcal{E}_\gamma(u,u).$$

This is equivalent to

$$(x-b)^2 \leqslant 2ab + b^2 \implies x \leqslant b + \sqrt{2ab+b^2} \leqslant 2b + \sqrt{2ab}.$$

Plugging in a, b, x yields (3.7).

■

We use Lemma 3.5 to show that $\mathcal{E}^\alpha \xrightarrow{\alpha\to\infty} \mathcal{E}$ on $\mathcal{F}\times\mathcal{F}$.

sf-35 **3.5 Theorem** (Resolvent to Form). *Let $u, v \in L^2(m)$ and $(\mathcal{E},\mathcal{F})$ be a closed form (i.e. bilinear, $(\mathcal{E}_1) - (\mathcal{E}_3)$ hold true). Then*

(i) $u \in \mathcal{F} \iff \limsup_{\alpha\to\infty} \mathcal{E}^\alpha(u,u) < \infty \iff \limsup_{\alpha\to\infty} |\mathcal{E}^\alpha(u,u)| < \infty$.

(ii) $u, v \in \mathcal{F} \implies \lim_{\alpha\to\infty} \mathcal{E}^\alpha(u,v) = \mathcal{E}(u,v)$.

(iii) $u \in \mathcal{F}$, $\alpha > \gamma \implies \mathcal{E}_\beta(\alpha\mathbf{G}_\alpha u - u, \alpha\mathbf{G}_\alpha u - u) \xrightarrow{\alpha\to\infty} 0$, *i.e.* $\alpha\mathbf{G}_\alpha u \xrightarrow{\alpha\to\infty} u$ *in* $(\mathcal{F}, \mathcal{E}^s_\beta)$. *Note that* $\alpha\mathbf{G}_\alpha u \xrightarrow{\alpha\to\infty} u$ *is clear by the strong continuity of* $(\mathbf{G}_\alpha)_{\alpha>0}$.

Proof. (i) By (**??**), we get for all $u \in \mathcal{F}$

$$\begin{aligned}
\mathcal{E}^\alpha(u,u) &\overset{??}{=} \mathcal{E}_\alpha(\alpha\mathbf{G}_\alpha u, u) - \alpha \langle \alpha\mathbf{G}_\alpha u, u\rangle_{L^2} \\
&= \alpha \langle u, u\rangle_{L^2} - \alpha \langle \alpha\mathbf{G}_\alpha u, u\rangle_{L^2} \\
&\overset{\text{CSI}}{\geqslant} \alpha \|u\|_{L^2}^2 - \alpha \left\|\alpha\mathbf{G}_\alpha u\right\|_{L^2} \|u\|_{L^2} \\
&\geqslant \alpha \|u\|_{L^2} - \alpha \frac{\alpha}{\alpha-\gamma} \|u\|_{L^2} \\
&= -\frac{\alpha\,\gamma}{\alpha-\gamma} \|u\|_{L^2}^2 .
\end{aligned}$$

Hence

$$\liminf_{\alpha\to\infty} \mathcal{E}^\alpha(u,u) \geqslant -\gamma \|u\|_{L^2}^2 > -\infty.$$

This gives the second «$\iff$» in (i). Now let $u \in \mathcal{F}$. By (3.7),

$$\mathcal{E}^\alpha(u,u) \overset{(3.7)}{\leqslant} \kappa^2 \mathcal{E}_\gamma(u,u)\sqrt{\gamma}\,\frac{\alpha}{\alpha-\gamma} \|u\|_{L^2} \sqrt{\mathcal{E}_\gamma(u,u)} < \infty$$

uniformly for all $\alpha > \gamma$. It remains to prove $\Longleftarrow$. Assume that $\limsup_\alpha \mathcal{E}^\alpha(u,u) < \infty$. For $\beta > \gamma$, we have by (3.5)

$$\begin{aligned}\mathcal{E}_\beta(\alpha\mathbf{G}_\alpha u, \alpha\mathbf{G}_\alpha u) &= \mathcal{E}(\alpha\mathbf{G}_\alpha u) + \beta\left\|\alpha\mathbf{G}_\alpha u\right\|_{L^2} \\ &\overset{(3.5)}{\leqslant} \mathcal{E}^\alpha(u) + \beta\frac{\alpha}{\alpha-\gamma}\left\|u\right\|_{L^2}^2 .\end{aligned}$$

Thus,

$$\limsup_{\alpha\to\infty} \mathcal{E}_\beta(\alpha\mathbf{G}_\alpha u) \leqslant \limsup_{\alpha\to\infty} \mathcal{E}^\alpha(u) + \beta\left\|u\right\|_{L^2}^2 .$$

Consequently, $(\alpha\mathbf{G}_\alpha u)_{\alpha>\gamma}$ is bounded in the Hilbert space $(\mathcal{F}, \mathcal{E}^s_\beta)$. Therefore, there exists a subsequence which converges weakly[5]:

$$\alpha_n\mathbf{G}_{\alpha_n} u \overset{\mathcal{E}^s_\beta}{\longrightarrow} v \in \mathcal{F}.$$

By strong continuity, $\alpha_n\mathbf{G}_{\alpha_n} u \to u$ in L^2. Hence $u = v \in \mathcal{F}$.

(ii) Using (iii), we obtain

$$\mathcal{E}^\alpha(u,v) \overset{(??)}{=} \mathcal{E}_\beta(\alpha\mathbf{G}_\alpha u, v) - \beta\left\langle\alpha\mathbf{G}_\alpha u, v\right\rangle_{L^2} \overset{\alpha\to\infty}{\dashrightarrow} \mathcal{E}_\beta(u,v) - \beta\left\langle u, v\right\rangle_{L^2} = \mathcal{E}(u,v).$$

(iii) **Aim** $\quad \alpha\mathbf{G}_\alpha u \overset{n\to\infty}{\dashrightarrow} u$ in $(\mathcal{F}, \mathcal{E}^s_\beta)$. As in Theorem **??**), we set

$$L := (\gamma+\varepsilon) - \mathbf{G}^{-1}_{\gamma+\varepsilon}, \;\; \mathcal{D}(L) := \mathbf{G}_{\gamma+\varepsilon}(L^2).$$

Assume, for a moment, that $\mathcal{D}(L) \subset \mathcal{F}$ is dense under $\mathcal{E}^s_\beta$.

(a) For $u \in \mathcal{D}(L)$ there exists $f \in L^2(m)$ such that $u = \mathbf{G}_\alpha f$. By (**??**) and the strong continuity of $(\mathbf{G}_\alpha)_{\alpha>0}$,

$$\begin{aligned}\mathcal{E}_\beta(\alpha\mathbf{G}_\alpha \underbrace{\mathbf{G}_\beta f}_{=u} - \underbrace{\mathbf{G}_\beta f}_{=u}) &= \mathcal{E}_\beta(\mathbf{G}_\beta(\alpha\mathbf{G}_\alpha f - f), \alpha\mathbf{G}_\alpha u - u) \\ &= \left\langle\alpha\mathbf{G}_\alpha f - f, \alpha\mathbf{G}_\alpha u - u\right\rangle_{L^2} \overset{\alpha\to\infty}{\dashrightarrow} 0.\end{aligned}$$

(b) Now let $u \in \mathcal{F}$ arbitrary and $u \in \mathcal{D}(L)$. Then

$$\begin{aligned}\mathcal{E}_\beta(\alpha\mathbf{G}_\alpha u - \alpha\mathbf{G}_\alpha w) &\overset{(3.5)}{\leqslant} \mathcal{E}^\alpha(u-w, u-w) + \beta\left\|\alpha\mathbf{G}_\alpha(u-w)\right\|^2_{L^2} \\ &\leqslant \mathcal{E}^\alpha(u-w, u-w) + \beta\left(\frac{\alpha}{\alpha-\gamma}\right)^2\left\|u-w\right\|^2_{L^2} \\ &\overset{(3.7)}{\leqslant} \kappa^2\mathcal{E}_\gamma(u-w) + \kappa\sqrt{\gamma}\,\frac{\alpha}{\alpha-\gamma}\,\left\|u-w\right\|^2_{L^2}\,\sqrt{\mathcal{E}_\gamma(u-w)} \\ &\leqslant C\;\mathcal{E}_{\beta'}(u-w, u-w)\end{aligned}$$

---

[5]Weak convergence in $(\mathcal{F}, \mathcal{E}^s_\beta)$ means $\forall w \in \mathcal{F} : \mathcal{E}^s_\beta(\alpha_n\mathbf{G}_{\alpha_n} u - v, w) \overset{n\to\infty}{\longrightarrow} 0$

for some $\beta' > \beta$ and $C > 0$. Consequently, if u is close to w, then $\alpha \mathbf{G}_\alpha u$ is close to $\alpha \mathbf{G}_\alpha w$. This implies that (!) holds on $\mathcal{F}$, not only on $\mathcal{D}(L)$.

(c) $\mathbf{G}_\beta(L^2)$ is $\mathcal{E}^s_\beta$-dense in $\mathcal{F}$ for all $\beta > \gamma$ is Lemma 3.6.

■

sf-36 **3.6 Lemma.** *$\mathcal{D}(L) = \mathbf{G}_\beta(L^2)$ is dense in $\mathcal{F}$ under $\mathcal{E}^s_\beta$, $\beta > \gamma$.*

Proof. Let $u \in \mathcal{F}$. As in the proof of Theorem 3.5 (i), we find that $(\mathcal{E}_\beta(\alpha \mathbf{G}_\alpha u))_{\alpha\in\mathbb{N}}$ is bounded. This means that $(\alpha \mathbf{G}_\alpha u)_{\alpha\in\mathbb{N}}$ is bounded in $(\mathcal{F}, \mathcal{E}^s_\beta)$. Since this is a Hilbert space, there exists a subsequence $(u_k)_{k\in\mathbb{N}}$ of $(\alpha \mathbf{G}_\alpha u)_{\alpha\in\mathbb{N}}$ such that $u_k \to u$, i.e.

$$\mathcal{E}^s_\beta(u_k - u, w) \xrightarrow{k\to\infty} 0 \quad \forall w \in \mathcal{F}.$$

We need strong $\mathcal{E}^s_\beta$-limit. Use a Banach-Saks argument:

(i) $m_1 := 1$. Find $m_2 > m_1$ such that

$$\mathcal{E}^s_\beta(u_{m_2} - u, \underbrace{u_{m_1} - u}_{\hat{=}\, w}) \leqslant 1.$$

(ii) $m_1 < \ldots < m_k$ are chosen. Pick $m_{k+1} > m_k$ with

$$\mathcal{E}^s_\beta(u_{m_{k+1}} - u, u_{m_l} - u) \leqslant \frac{1}{k}, \quad \text{for all } l = 1, \ldots, k.$$

Ok, b/o weak convergence.

Then

$$\begin{aligned}
\mathcal{E}^s_\beta\left(u - \frac{u_{m_1} + \ldots + u_{m_k}}{k}\right) &= \mathcal{E}^s_\beta\left(\sum_{i=1}^k \frac{u - u_{m_i}}{k}\right) \\
&= \frac{1}{k^2} \sum_{i=1}^k \underbrace{\mathcal{E}^s_\beta(u - u_{m_i})}_{\text{uniformly bdd}} + \frac{2}{k^2} \sum_{i<j\leqslant k} \mathcal{E}^s_\beta(u - u_{m_i}, u - u_{m_j}) \\
&\leqslant \frac{C}{k^2} k + \frac{2}{k^2} \sum_{j=1}^k \frac{j-1}{j} \\
&\leqslant \frac{C'}{k} \xrightarrow{k\to\infty} 0,
\end{aligned}$$

i.e. the Cèasaro means converge strongly. ■

Aim characterize sub-Markovianity $\begin{cases} (T_t)_{t\geqslant 0} \\ (\alpha \mathbf{G}_\alpha)_{\alpha>\gamma} \end{cases}$ via $(\mathcal{E}, \mathcal{F})$

sf-37 **3.7 Definition.** Let $(\mathcal{E}, \mathcal{F})$, $\mathcal{F} \subset L^2(X,m)$ a bilinear form. **A lower bounded semi-Dirichlet form** (SDF_γ) is a closed form (cf. **??**), i.e.

$$\exists \gamma \geqslant 0 : \mathcal{E}_\gamma(u) = \mathcal{E}(u) + \gamma \langle u, u \rangle_{L^2} \geqslant 0 \qquad (\mathcal{E}_1)$$ sf::eqe1

$$\exists \kappa \geqslant 0 : |\mathcal{E}(u,v)| \leqslant \kappa \sqrt{\mathcal{E}_\gamma(u)} \sqrt{\mathcal{E}_\gamma(v)} \qquad (\mathcal{E}_2)$$ sf::eqe2

$$(\mathcal{F}, \mathcal{E}^s_\alpha(\cdot,\cdot))_{\alpha>\gamma} \text{ Hilbert spaces,} \qquad (\mathcal{E}_3)$$ sf::eqe3

satisfying in addtion,

$$\forall u \in \mathcal{F}\, \forall b \geqslant 0 : u \wedge b \in \mathcal{F} \text{ and } \mathcal{E}(u \wedge b, u) \geqslant \mathcal{E}(u \wedge b, u \wedge b), \qquad (\mathcal{E}_4)$$ sf::eqe4

SDF_0 (i.e. $\gamma = 0$) are the positive semi-DF. Any SDF_0 so that

$$\forall u \in \mathcal{F}\, \forall b \geqslant 0 : u \wedge b \in \mathcal{F} \text{ and } \mathcal{E}(u, u \wedge b) \geqslant \mathcal{E}(u \wedge b, u \wedge b) \qquad (\hat{\mathcal{E}}_4)$$ sf::eqe4

is called **non-symmetric** Dirichlet form. If $\mathcal{E}(u,v) = \mathcal{E}(v,u)$ it is a **(symmetric) Dirichlet form**.

Here SDF_γ is main object.

sf-38 **3.8 Remark.** (a) In $(\mathcal{E}_4)$, $(\hat{\mathcal{E}}_4)$ the claim $u \wedge b \in \mathcal{F}$ is essential! Means: $\mathcal{F}$ has lattice structure, i.e. $u \in \mathcal{F} \Longrightarrow u^+, u^-, |u|, (-n) \vee u \wedge n \in \mathcal{F}$. Note, this implication is ok for Lipschitz-functions, but not for C_c^∞, C^1.[6]

(b) Rôle of symmetry in $(\mathcal{E}_4)$. If $\mathcal{E}$ is symmetric, then $(\mathcal{E}_4) \Longleftrightarrow (\hat{\mathcal{E}}_4) \Longleftrightarrow \forall u \in \mathcal{F} : |u| \in \mathcal{F}$ and $\mathcal{E}(|u|) \leqslant \mathcal{E}(u)$.

sf-39 **3.9 Proposition.** *Let $(\mathcal{E}, \mathcal{F})$ be a closed form, resolvent $(\mathbf{G}_\alpha)_{\alpha>\gamma}$. TFAE:*

$$\forall u \in \mathcal{F}\, \forall b \geqslant 0 : u \wedge b \in \mathcal{F} \text{ and } \mathcal{E}(u \wedge b, u) \geqslant \mathcal{E}(u \wedge b) \qquad (\mathcal{E}_4)$$ sf::eqe4

$$\forall u \in \mathcal{F} : u^+ \wedge 1 \in \mathcal{F} \text{ and } \mathcal{E}(u^+ \wedge 1, u) \geqslant \mathcal{E}(u^+ \wedge 1) \qquad (\mathcal{E}_4')$$ sf::eqe4

$$\forall u \in \mathcal{F} : u^+ \wedge 1 \in \mathcal{F} \text{ and } \mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geqslant -\gamma \left\| u - u^+ \wedge 1 \right\|_{L^2}^2 \qquad (\mathcal{E}_4'')$$ sf::eqe4

$$\forall f \in L^2,\ 0 \leqslant f \leqslant 1 \ m\text{-a.e.} : 0 \leqslant \alpha \mathbf{G}_\alpha f \leqslant 1 \ m\text{-a.e.}\ (\alpha > \gamma) \qquad \text{(RM)}$$ sf::RM

[6] Exercise: $D = \mathbb{R}^d$, $\mathcal{E}(u,v) = \int_D \nabla u \, \nabla v \mathrm{d}x$, cf. **??**. Prove: $\mathcal{F} = W'(D)$, then $u \in \mathcal{F} \Longrightarrow |u| \in \mathcal{F}$ and $\mathcal{E}(|u|) \leqslant \mathcal{E}(u)$. Idea for the implication is, to find $\frac{\partial}{\partial x_i} |u|$ (weak) and then for φ Lipschitz: $\frac{\partial}{\partial x_i} \varphi \circ u = ?$. Look at Gilbarg + Trudinger: Elliptic PDEs of 2nd order, p. 150-155, §7.4

Proof. $(\mathcal{E}_4) \implies (\mathcal{E}_4')$ Take $u \in \mathcal{F} \overset{(\mathcal{E}_4)}{\implies} u^+ = -[(-u)\wedge 0] \in \mathcal{F} \overset{(\mathcal{E}_4)}{\implies} u^+ \wedge 1 = (u \wedge 1)^+ = -\{[-(u\wedge 1)]\wedge 0\} \in \mathcal{F}$ or obviously by 1st line. Then

$$\begin{aligned}
\mathcal{E}(u^+ \wedge 1, u) &= \mathcal{E}(u^+ \wedge 1, \underbrace{u - u\wedge 1}_{=u^+ - u^+\wedge 1}) + \mathcal{E}(u^+ \wedge 1, u \wedge 1)\\
&= \underbrace{\mathcal{E}(u^+ \wedge 1, u - u \wedge 1)}_{\geqslant 0 \text{ by } (\mathcal{E}_4)} + \mathcal{E}(+\{[-(u\wedge 1)]\wedge 0\}, -[u \wedge 1])\\
&\overset{(\mathcal{E}_4)^7}{\geqslant} 0 + \mathcal{E}(-[-(u\wedge 1)]\wedge 0, -[-(u\wedge 1)]\wedge 0)\\
&= \mathcal{E}(u^+ \wedge 1, u^+ \wedge 1)
\end{aligned}$$

$(\mathcal{E}_4') \implies (\mathcal{E}_4'')$ We have

$$\begin{aligned}
\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) &= \mathcal{E}(u - u^+ \wedge 1, u - u^+ \wedge 1) + \underbrace{2\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1)}_{\geqslant 0,\ (\mathcal{E}_4')}\\
&\overset{(\mathcal{E}_1)}{\geqslant} -\gamma \left\|u - u^+ \wedge 1\right\|_{L^2}^2
\end{aligned}$$

$(\mathcal{E}_4'') \implies$ (RM) Take $f \in L^2(m)$, $0 \leqslant f \leqslant 1$ m-a.e. Set $u = \alpha \mathbf{G}_\alpha f \in \mathcal{F}$ $(\alpha > \gamma)$.

Aim $u \overset{\text{a.e.}}{=} u^+ \wedge 1$.

$$\begin{aligned}
(\mathcal{E}_4'') &\implies \gamma \left\|u - u^+ \wedge 1\right\|_{L^2}^2 \geqslant -\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1)\\
(\mathcal{E}_1) &\implies \gamma \left\|u - u^+ \wedge 1\right\|_{L^2}^2 \geqslant -\mathcal{E}(u - u^+ \wedge 1, u - u^+ \wedge 1)
\end{aligned}$$

So add both with factor $\frac{1}{2}$

$$\begin{aligned}
\gamma \left\|u - u^+ \wedge 1\right\|_{L^2}^2 &\geqslant -\mathcal{E}_\alpha(u, u - u^+ \wedge 1) + \alpha \left\langle u, u - u^+ \wedge 1\right\rangle_{L^2}\\
&\overset{u = \alpha \mathbf{G}_\alpha 4f}{\underset{\mathcal{E}_\alpha(\mathbf{G}_\alpha f, \cdot) = \langle f, \cdot\rangle_{L^2}}{=}} \alpha \left\langle u - f, u - u^+ \wedge 1\right\rangle_{L^2}\\
&= \alpha \left\|u - u^+ \wedge 1\right\|_{L^2}^2 + \underbrace{\alpha \left\langle u^+ \wedge 1 - f, u - u^+ \wedge 1\right\rangle_{L^2}}_{\geqslant 0, \text{ see } \star \text{ below}}\\
&\implies (\alpha - \gamma) \left\|u - u^+ \wedge 1\right\|_{L^2}^2 \leqslant 0\\
&\implies \|\cdot\|^2 = 0\\
&\overset{\text{a.e.}}{\implies} u = u^+ \wedge 1\\
&\implies \text{(RM) for } u \geqslant 1
\end{aligned}$$

To see $\star$:

$$u - u^+ \wedge 1 = \begin{cases} u - 1, & u \geqslant 1\\ 0, & \text{else}\\ u, & u \geqslant 0 \end{cases}$$

Now

$$
\begin{aligned}
\langle u^+ \wedge 1 - f, u - u^+ \wedge 1\rangle_{L^2(m)} &= \int \left(u^+ \wedge 1 - f\right)\left(u - u^+ \wedge 1\right) \mathrm{d}m \\
&= \int_{u\geqslant 1} (\underbrace{u^+ \wedge 1}_{=1,\ \geqslant 0,\ f\leqslant 1} - f)\underbrace{(u-1)}_{\geqslant 0} \mathrm{d}m + \int_{u\leqslant 0} \underbrace{(\overbrace{u^+ \wedge 1}^{=0} - f)}_{=\ -f\leqslant 0} \underbrace{u}_{\leqslant 0} \mathrm{d}m \\
&\geqslant 0
\end{aligned}
$$

(RM) $\Longrightarrow$ ($\mathcal{E}_4$) Take $u \in \mathcal{F}$, $b \geqslant 0$.

Know $\alpha \mathbf{G}_\alpha(\underbrace{u \wedge b}_{\in L^2}) \leqslant b$ b/o (RM)

$$
\begin{aligned}
&\Longrightarrow & \alpha\mathbf{G}_\alpha(u\wedge b) - (u\wedge b) &\leqslant b - (u \wedge b) \\
&\overset{\cdot(u\wedge b - u)}{\Longrightarrow} & (u\wedge b - \alpha\mathbf{G}_\alpha(u\wedge b))(u\wedge b - u) &\leqslant (u\wedge b - u)(u \wedge b - b) \\
&\overset{\int\ldots\mathrm{d}m}{\Longrightarrow} & \alpha\,\langle u\wedge b - \alpha\mathbf{G}_\alpha(u\wedge b), u\rangle_{L^2} &\geqslant \alpha\,\langle u\wedge b - \alpha\mathbf{G}_\alpha(u\wedge b), u\wedge b\rangle_{L^2} \\
&\underset{\text{Thm 3.5}}{\overset{\text{Def of }\mathcal{E}^\alpha}{\Longrightarrow}} & \mathcal{E}^\alpha(u\wedge b, u) &\geqslant \mathcal{E}^\alpha(u\wedge b, u\wedge b).
\end{aligned}
$$

If (!) $u \wedge b \in \mathcal{F}$, then $\lim_{\alpha\to 0}\mathcal{E}^\alpha = \mathcal{E}$ and ($\mathcal{E}_4$) follows. But this follows with 3.5 (a). We need

$$
\limsup_{\alpha\to\infty} \mathcal{E}^\alpha(u\wedge b, u\wedge b) < \infty. \tag{**}
$$

So

$$
\begin{aligned}
\mathcal{E}^\alpha(u\wedge b, u\wedge b) &\overset{\text{above}}{\leqslant} \mathcal{E}^\alpha(u\wedge b, u) \\
&\overset{\text{L3.4, (8)}}{\leqslant} \kappa\sqrt{\mathcal{E}^\alpha_\lambda(u\wedge b, u\wedge b)}\sqrt{\mathcal{E}_\gamma(u,u)}, \quad \lambda \geqslant \gamma\left(\alpha\frac{\alpha}{\alpha-\gamma}\right)^2.
\end{aligned}
$$

If $x^2 = \mathcal{E}^\alpha_\lambda(u\wedge b, u\wedge b)$. This is an inequality of the type

$$
x^2 \leqslant \overline{\kappa}\,|x| + c_\lambda^2 \Longrightarrow |x| \text{ is bounded}
$$

The idea: complete the square $\Longrightarrow |x| \leqslant \frac{c_\lambda}{\sqrt{1+\frac{1}{2}\overline{\kappa}^2 - \frac{1}{2}\overline{\kappa}}} \Longrightarrow$ (**) is true.

■

Pass on to semigroups. Next is a corollary to Hille-Yosida.

sf-310 **3.10 Corollary.** *$(T_t)_t$ is (C_0)-contraction semigroup on $\mathcal{B} = L^2(m)$. Let $U_\alpha = \mathbf{G}_\alpha$, $\alpha > 0$, be its resolvent. TFAE:*

$$
f \in L^2(m),\ 0 \leqslant f \leqslant 1 \Longrightarrow 0 \leqslant T_t \leqslant 1 \quad \forall t \geqslant 0 \tag{SM}
$$
sf::eqSM

$$
f \in L^2(m),\ 0 \leqslant f \leqslant 1 \Longrightarrow 0 \leqslant \alpha U_\alpha f \leqslant 1 \quad \forall \alpha > 0 \tag{RM}
$$
sf::eqRM

Proof. (SM) $\Longrightarrow$ (RM)

$$\alpha U_\alpha f(x) = \int_0^\infty \alpha e^{-\alpha t} \underbrace{T_t f(x)}_{\in[0,1]} \mathrm{d}t \in \left[0, \int_0^\infty \alpha e^{-\alpha t} \mathrm{d}t\right] = [0,1]$$

(RM) $\Longrightarrow$ (SM) In the proof of Hille-Yosida we had

$$\forall u \in \mathfrak{D}(\mathbf{A}),\ 0 \leqslant u \leqslant 1 : T_t^{(\alpha)} u = e^{-t\alpha} \sum_{n=0}^\infty \frac{(\alpha t)^n}{n!} \underbrace{(\alpha U_\alpha)^n u}_{\in[0,1]} \in [0,1]$$

So for $u \in \mathfrak{D}(\mathbf{A})$, pf. of Hille-Yosida, we get

$$T_t u(x) = \lim_{\substack{\alpha\to\infty \\ n\to\infty}} T_t^{(\alpha)} u(x) \in [0,1] \quad \text{in } L^2 \text{ a.e. sub-sequence.},$$

so (SM) ok on $\mathfrak{D}(\mathbf{A})$. Let $f \in L^2$, $0 \leqslant f \leqslant 1$. Fix $\lambda > 0$, then by (RM) $\lambda U_\lambda f(x) \in [0,1]$ and $\lambda U_\lambda f \in \mathfrak{D}(\mathbf{A})$.

$\Longrightarrow 0 \leqslant T_t \lambda U_\lambda f \leqslant 1$

$\Longrightarrow 0 \leqslant \lambda U_\lambda T_t f \leqslant 1$ and $\lambda U_\lambda \to \mathrm{id}$

$\overset{\lambda\to\infty}{\Longrightarrow} 0 \leqslant T_t f \leqslant 1.$ ∎

sf-311 **3.11 Remark.** 3.10 remains valid for the semigroup $e^{t\beta} T_t$, $(T_t)_t$ contraction semigroup, and the resolvent $(U_\alpha)_{\alpha>\beta}$. Hence, we can apply 3.10 in 3.9.

$T : L^2(m) \to L^2(m)$, T **positive (positivity preserving)**, if $f \geqslant 0$ a.e. $\Longrightarrow Tf \geqslant 0$ a.e.

Consequences

- $f \leqslant g \Longrightarrow Tf \leqslant Tg$ (take $0 \leqslant g - f \oplus T$ linear $\oplus T$ positive)
- $\pm f \leqslant |f| \Longrightarrow \pm Tf = T(\pm f) \leqslant T|f| \Longrightarrow |Tf| \leqslant T|f|$
- $T : L^2 \to L^2$ is continuous. As T is linear:

$$T \text{ cts } \Longleftrightarrow T \text{ bounded } \Longleftrightarrow \exists c\ \forall f : \|Tf\|_{L^2} \leqslant c\,\|f\|_{L^2}$$

sf-312 **3.12 Lemma.** *$T : L^2(m) \to L^2(m)$ positive (linear) operator, i.e.*

$$\forall f \in L^2\ f \geqslant 0 : Tf \geqslant 0 \tag{11}$$ sf::eq11

then T is continuous, i.e. there exists $c > 0$ such that for all $f \in L^2(m)$:

$$\|Tf\|_{L^2} \leqslant c\,\|f\|_{L^2} \tag{12}$$ sf::eq12

Proof. 1^o linear + (11) $\Longrightarrow$ monotone.

2^o Δ-inequality $\pm f \leqslant |f| \overset{\text{monotone}}{\Longrightarrow} \pm Tf \leqslant T|f| \Longrightarrow |Tf| \leqslant T|f|$

3^o Continuity. Assume (12) fails

$$\exists (f_n)_n \subset L^2,\ \|f_n\|_{L^2} \leqslant 1,\ \|Tf_n\| \geqslant 4^n,$$

and set $\varphi := \sum_{n=1}^{\infty} 2^{-n} |f_n| \in L^2(m)$.

$\Longrightarrow T\varphi \overset{\text{monotone}}{\geqslant} 2^{-n} T|f_n| \overset{2^o}{\geqslant} 2^{-n}|Tf_n|\ \forall n$

$\Longrightarrow \|T\varphi\|_{L^2} \geqslant 2^{-n}\|Tf_n\|_{L^2} \geqslant 2^n \uparrow \infty \not\downarrow T\varphi \in L^2$

■

sf-313 **3.13 Lemma.** *$T : L^2(m) \to L^2(m)$ continuous, sub-Markov, i.e.*

$$\forall f \in L^2,\ 0 \leqslant f \leqslant 1 : 0 \leqslant Tf \leqslant 1. \tag{13}$$ sf::eq13

Then f is positive and extends to L^∞ (and is positive, sub-Markov and contraction on L^∞, uniqueness?).

Note that this answers a step in an earlier proof: $\alpha \mathbf{G}_\alpha(u \wedge b) \leqslant b$. DIY (use monotone property and conclude this guy is positive).

Proof. 1^o $f \in L^2 \cap L^\infty(m),\ f \geqslant 0 \Longrightarrow 0 \leqslant \frac{f}{\|f\|_\infty} \leqslant 1$

$\Longrightarrow 0 \leqslant Tf \leqslant \|f\|_\infty$ by (13)

2^o **Claim** $f_n \in L^\infty,\ f_n \geqslant 0,\ f_n \uparrow f \in L^2 \Longrightarrow Tf_n \uparrow Tf$[8]

$1^o \Longrightarrow Tf_m \leqslant Tf_n\ \forall m \leqslant n$ (note: $f_n \in L^2$!)

$\overset{\text{cts}}{\Longrightarrow} \|Tf - Tf_n\|_{L^2} \leqslant c\|f - f_n\|_{L^2} \xrightarrow[\text{conv.}]{\text{mono}} 0$

$Tf_{n(k)} \xrightarrow{a.e.} Tf$

$\overset{\text{increasing}}{\Longrightarrow} Tf_n \uparrow Tf$ (full sequence!)

3^o $f \in L^2_+$[9] Then $f_n := f \wedge n \in L^2_+ \cap L^\infty_+$, and so, $Tf = \sup_n Tf_n \geqslant 0$

4^o $f \in L^\infty_+,\ f_n, g_n \in L^2_+,\ f_n, g_n \uparrow f$[10]

$$Tg_n \uparrow T(g_n \wedge f_m) \leqslant Tf_m \leqslant \sup_m Tf_m$$

[8] Kind of «Daniell extension».
[9] «+» means $f \geqslant 0$ m-a.e.
[10] Not trivial, e.g. by our topological assumptions on X and m $\exists B_n \uparrow X$, $\overline{B_n}$ cpt. and $m(B_n) < \infty$, so $f_n := f \mathbb{1} B_n$ is good.

$\Longrightarrow \sup_n Tg_n = \sup_n T(g_n \wedge f_m) \leqslant \sup_m Tf_m$
$\xrightarrow[g_n \leftrightarrow f_n]{\text{sym.}} \sup_n Tg_n = \sup_m Tf_m =: Tf$
By linearity $Tf := Tf^+ - Tf^- \; \forall f \in L^\infty$ positive, (13), L^∞-contraction clear. ■

sf-314 **3.14 Corollary** (to 3.9, 3.10). *The conditions* $(\mathcal{E}_4)-(\mathcal{E}_4'')$ *and (RM) and (SM) are equivalent to*

$$(\hat{T}_t)_{t\geqslant 0} \text{ extends to a contraction semigroup on } L'(m) \text{ and } \hat{T}_t \text{ positive.} \qquad (14)$$ sf::eq14

Proof. (SM) $\Longrightarrow$ (**??**). (T_t) is also L^∞-extension form 3.13. Then $f, g \in L^2$:

$$|\langle T_t f, g\rangle_{L^2}| = \left|\left\langle f, \hat{T}_t g\right\rangle_{L^2}\right| \leqslant \|T_t f\|_{L^\infty} \|g\|_{L^1}$$

$\Longrightarrow \left\|\hat{T}_t g\right\|_{L^1} = \sup_{f\in L^\infty} \frac{\langle T_t f, g\rangle}{\|f\|_{L^\infty}} \leqslant \|g\|_{L^1}$
$\Longrightarrow \hat{T}_t$ is an L^1-contraction.
Clear $\hat{T}_t$ inherits semigroup property of T_t (first equality).

Positivity $B_n \in \mathcal{B}(X)$, $B_n \uparrow X$, $m(B_n) < \infty$. Set $A_n := B_n \cap \left\{\hat{T}_t g < 0\right\}$, $g \in L^1_+$, g fixed. Then

$$\begin{aligned} 0 \geqslant \int_{B_n \cap \{\hat{T}_t g < 0\}} \hat{T}_t g \mathrm{d}m &\overset{\text{def}}{=} \left\langle \mathbb{1}_{A_n}, \hat{T}_t g\right\rangle_{L^2} \\ &= \left\langle T_t \mathbb{1}_{A_n}, g\right\rangle_{L^2} \geqslant 0 \end{aligned}$$

$\Longrightarrow$ «0» $\Longrightarrow m\left\{\hat{T}_t g < 0\right\} = 0$. The converse is exactly the same type of argument. ■

sf-315 **3.15 Remark.** $(\hat{T}_t)_{t\geqslant 0}$, $(\hat{\mathbf{G}}_\alpha)_{\alpha>\gamma}$ are sub-Markov, if $(\hat{\mathcal{E}}_4)$ holds.[11]

Two more technical things

(a) 3.16 $(\mathcal{E}_4)$ under closure,

(b) 3.17 stability of $\mathcal{F}$ under Lipschitz-maps.

sf-316 **3.16 Proposition.** $(\mathcal{E}, \mathcal{F})$ *is closable, lower bounded* (γ) *bilinear form, sectorial* $(\mathcal{E}_2)$ *and enjoys* $(\mathcal{E}_4)$*. Then its closure* $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ *has* $(\mathcal{E}_4)$*, i.e. it is a* SDF_γ.

Proof. $u \in \overline{\mathcal{F}}$, $(u_n) \subset \mathcal{F} \subset \overline{\mathcal{F}}$ and $\overline{\mathcal{E}}_\lambda(u-u_n, u-u_n) \xrightarrow{n\uparrow\infty} 0$ $(\lambda > \gamma$ fixed). Then clearly

$$\overline{\mathcal{E}}_\lambda = \underbrace{\overline{\mathcal{E}}_\gamma}_{\geqslant 0} + (\lambda-\gamma)\langle\cdot,\cdot\rangle_{L^2},$$

[11] reason: all in 2^{nd} argument of $\mathcal{E}(\star, \cdot)$

so $u_n \xrightarrow{L^2(m)} u$ and by (DOM) $u_n \wedge b \xrightarrow{L^2(m)} u \wedge b$. Moreover,

$$\begin{aligned}\overline{\mathcal{E}}(u_n \wedge b, u_n \wedge b) &\overset{(\mathcal{E}_4)}{\leqslant} \overline{\mathcal{E}}(u_n \wedge b, u_n)\\ &\overset{(\mathcal{E}_2)}{\leqslant} \kappa\sqrt{\mathcal{E}_\lambda(u_n \wedge b, u_n \wedge b)}\sqrt{\overline{\mathcal{E}}_\lambda(u_n, u_n)}.\end{aligned}$$

As in the last step of the proof of Theorem 3.9, we get

$$\sup_n \overline{\mathcal{E}}(u_n \wedge b, u_n \wedge b) \leqslant c_\lambda < \infty.$$

Since $\mathcal{E}_\lambda$ is a scalar product on a Hilbert space $\overline{\mathcal{F}}$, there exists $(u_{n(k)})_k \subset (u_n)_n$: $u_{n(k)} \wedge b \underset{w}{\xrightarrow{\overline{\mathcal{F}}}} u \wedge b$ and $u \wedge b \in \overline{\mathcal{F}}$.[12]

$\Longrightarrow u \wedge b \in \overline{\mathcal{F}}$. Now use «resonance theorem» for Hilbert spaces (weak convergence $\Longrightarrow$ Fatou)

$$\overline{\mathcal{E}}_\lambda(u \wedge b, u \wedge b) \leqslant \liminf \overline{\mathcal{E}}_\lambda(u_{n(k)} \wedge b)$$

Since $u_{n(k)} \wedge b \xrightarrow{L^2(m)} u \wedge b$ we get

$$\begin{aligned}\overline{\mathcal{E}}(u \wedge b, u \wedge b) &\leqslant \liminf \overline{\mathcal{E}}(u_{n(k)} \wedge b)\\ &\leqslant \liminf \overline{\mathcal{E}}(u_{n(k)} \wedge b, u_{n(k)} - u + u)\\ &= \liminf \overline{\mathcal{E}}(u_{n(k)} \wedge b, u_{n(n)} - u) + \underbrace{\overline{\mathcal{E}}(u \wedge b, u)}_{w\text{-conv.}}\end{aligned}$$

Aim Show 1st term on the rhS $= 0$.

$$\left|\overline{\mathcal{E}}(u_{n(k)} \wedge b, u_{n(k)} - u)\right| \overset{(\mathcal{E}_2)}{\leqslant} \kappa \underbrace{\sqrt{\overline{\mathcal{E}}_\lambda(u_{n(k)} \wedge b)}}_{\text{bdd, see above}} \underbrace{\sqrt{\overline{\mathcal{E}}_\lambda(u_{n(n)} - \blacksquare}}_{\xrightarrow{k\uparrow\infty} 0 \text{ b/o closure}}$$

sf-317 **3.17 Definition.** A **normal contraction** is a map $T : \mathbb{R}^n \to \mathbb{R}$ ($n \in \mathbb{N}$) (not necessarily linear) such that

$$T(0) = 0, \quad |T(x) - T(y)|^2 \leqslant \sum_{i=1}^n \left|x_i - y_i\right|^2 \tag{15}$$ sf::eq15

Then ist is clear that for the Euclidean norm in $\mathbb{R}^d$

$$|T(x)| \leqslant |x|.$$

[12] Weak convergence means $\overline{\mathcal{E}}_\lambda(u_{n(k)} \wedge b, w) \xrightarrow{k\uparrow\infty} \overline{\mathcal{E}}_\lambda(v, w)$, $\forall w \in \overline{\mathcal{F}}$, $v \in \overline{\mathcal{F}}$ the weak limit. Problem here: Why $v = u \wedge b$?

sf-318 **3.18 Theorem.** *Let $(\mathcal{E},\mathcal{F})$ be a* SDF_γ *and let T be a normal contraction (n as in the definition). Then*

(a) $u_1, u_2, \dots, u_n \in \mathcal{F} \Longrightarrow T(u_1, u_2, \dots, u_n) \in \mathcal{F}$

(b) $\mathcal{E}(T \circ u, T \circ u) \leqslant \sum_{i=1}^n \underbrace{\mathcal{E}(u_i, u_i)}_{\text{lives on the diag. in } \mathcal{F}^2}$

Remark In the symmetric DF case, (a) + (b) $\iff (\mathcal{E}_4)$, $(\hat{\mathcal{E}}_4)$.

Proof. B/o Theorem 3.5 it is enough to show

$$\mathcal{E}^\alpha(T \circ u) \leqslant \sum_{i=1}^n \mathcal{E}^\alpha(u_i), \tag{$\star$}$$

where $u_i \in \mathcal{F}$, $T \circ u = T(u_1, \dots, u_n)$.[13] $(\star)$ follows from

$$\langle (1-\alpha \mathbf{G}_\alpha) T \circ f, T \circ f \rangle_{L^2(m)} \leqslant \sum_{i=1}^n \langle (1-\alpha \mathbf{G}_\alpha) T \circ f_i, T \circ f_i \rangle_{L^2(m)}, \tag{$\star'$}$$

for any $f = (f_1, \dots, f_n) \in L^2 \times \dots \times L^2(m)$. Can even go to a dense subset of L^2, e.g. step functions. WLOG

$$f_i = \sum_{k=1}^N \alpha_{ik} \mathbb{1}_{A_k},$$

$i = 1, \dots, n$, $A_k \in \mathcal{B}(X)$ disjoint (same for all i), $m_k = m(A_k) < \infty$ (ok as $f_i \in L^2$), N independent of i and $\alpha_{ik} \in \mathbb{R}$.

Note $T(f) = \sum_{k=1}^N \underbrace{T(\alpha_{1k}, \alpha_{2k}, \dots, \alpha_{nk})}_{=:\ \tau_k} \mathbb{1}_{A_k}$ is again a step function. By definition

$$a_{kl} = \left\langle \alpha \mathbf{G}_\alpha \mathbb{1}_{A_k}, \mathbb{1}_{A_l} \right\rangle_{L^2(m)} = a_{lk}$$

provided that $\mathbf{G}_\alpha = \hat{\mathbf{G}}_\alpha$ = symmetric. May assume $\mathcal{E} = \mathcal{E}^s$ for proving (a) + (b) $\Longrightarrow \mathbf{G}_\alpha = \hat{\mathbf{G}}_\alpha$ by construction in Theorem **??**.

$$\begin{aligned} \langle (1-\alpha \mathbf{G}_\alpha) T(f), T(f) \rangle_{L^2(m)} &= \sum_{k,l=1}^N \tau_k \tau_l \underbrace{\left\langle (1-\alpha \mathbf{G}_\alpha) \mathbb{1}_{A_k}, \mathbb{1}_{A_l} \right\rangle_{L^2(m)}}_{= m_k \delta_{kl} - a_{kl}} \\ &= \sum\sum_{0 \leqslant k < l \leqslant N} \underbrace{a_{kl}}_{\geqslant 0 \text{ b/o sub-M.}} (\tau_k - \tau_l)^2 + \underbrace{\sum_{k=1}^N \left(m_k - \sum_{l=1}^N a_{kl} \right) \tau_k^2}_{\geqslant 0 \text{ b/o } \sharp} \end{aligned}$$

---

[13] Enough, since $w \in \mathcal{F} \iff \liminf_{\alpha \to \infty} \mathcal{E}^\alpha(w) < \infty$ and $\mathcal{E}^\alpha(w) \xrightarrow{\alpha \uparrow \infty} \mathcal{E}(w)\ \forall w \in \mathcal{F}$. Recall $\mathcal{E}^\alpha(w) := \alpha \langle (1-\alpha \mathbf{G}_\alpha) w, w \rangle_{L^2(m)}$

(♯):

$$\begin{aligned}\sum a_{kl} &= \sum_l \int_X \alpha \mathbf{G}_\alpha \mathbb{1}_{A_k} \mathbb{1}_{A_l} \mathrm{d}m \\ &= \int_X \alpha \mathbf{G}_\alpha \mathbb{1}_{A_k} \mathbb{1}_{\bigcup A_l} \mathrm{d}m \\ &\overset{\text{sym.}}{=} \int_X \mathbb{1}_{A_k} \alpha \mathbf{G}_\alpha \mathbb{1}_{\bigcup A_l} \mathrm{d}m \\ &\overset{\text{sub-M.}}{\leqslant} \int_X \mathbb{1}_{A_k} \mathbb{1} \mathrm{d}m = m_k.\end{aligned}$$

Since the coeffiences are positive, we can use Lipschitz character of T (encoded in the τ_k's). So above gets

$$\begin{aligned}&\leqslant \sum_{k\leqslant l}\sum a_{kl} \sum_{i=1}^n \left(\alpha_{ik} - \alpha_{il}\right)^2 + \sum_k \left(m_k - \sum_l a_{kl} \right) \sum_{i=1}^n \alpha_{ik}^2 \\ &= \sum_{i=1}^n \left(\sum_{k^l}\sum a_{kl}(\alpha_{ik} - \alpha_{il})^2 + \sum_k \left(m_k - \sum_l a_{kl} \right) \alpha_{ik}^2 \right).\end{aligned}$$

With the same argument as with $\mathcal{E}^\alpha(T \circ f)$ one gets

$$\sum_{i=1}^n \langle (1 - \alpha \mathbf{G}_\alpha) f_i, f_i \rangle_{L^2}.$$

This finishes the prove. ■

Question $\left(\mathcal{E}^\alpha, \mathfrak{F} = L^2(m)\right)$ is this a SDF_γ? $(\mathcal{E}_1), (\mathcal{E}_3), (\mathcal{E}_4)$ ✓, but $(\mathcal{E}_2)$?, Problem $\mathcal{E}^\alpha(f, v)$ for $f \in L^2,\ v \in \mathfrak{F}$ and $\mathcal{E}^\alpha(f, \alpha \mathbf{G}_\alpha g)$ (Lemma 3.4)

Chapter 4

REGULAR (SYMMETRIC) SDF_γ

Aim Integral formulae for DF_γ

Problem non-symmetric case

Setting

- (X, d) local compact, separable
- $C_c(X) = \left\{u : X \to \mathbb{R} \text{ cts}, \ \operatorname{spt} u = \overline{\{u \neq 0\}} \text{ cpt.}\right\}$ does X have many compact sets? *YES!*
  $\exists K_n$ cpt, $K_n \uparrow X$, Idea $X = \overline{\left\{x_i : n \in \mathbb{N}\right\}}^d$, then $\overline{B_{r_i}(x_i)} = \overline{\left\{x : d(x, x_i) < r_i\right\}}$ is WLOG cpt ($r_i \ll 1$), $K = \bigcup_{\text{finite}} \overline{B_{r_i}(x_i)}$
- Urysohn's lemma $K \subset U \subset \overline{U} \subset X$ (Idea cover K by finitely many $\overline{B_{r_i}(x_i)}$),

$$\exists \varphi_{K,U}(x) = \frac{d(x, U^c)}{d(x, K) + d(x, U^c)},$$

  $\varphi_{K,U} \in C_c$, $\varphi_{K,U}\big|_K = \mathbb{1}$, $\varphi_{K,U}\big|_{U^c} = 0$
  $\exists$ hope that $\mathcal{F}$ is «rich» ($\rightsquigarrow$ life as in $\mathbb{R}^n$ etc.)

rsdf-41 **4.1 Definition.** A SDF_γ $(\mathcal{E}, \mathcal{F})$ is **regular**,

(a) $C_c(X) \cap \mathcal{F}$ is $\mathcal{E}_\alpha$-dense (sym!) in $\mathcal{F}$ ($\alpha > \gamma$)

(b) $C_c(X) \cap \mathcal{F}$ is $\|\cdot\|_\infty$-dense in $C_c(X)$

Note that $\|u\|_\infty = \sup |u|$ vs $\|u\|_{L^\infty}$ «m-esssup».

rsdf-42 **4.2 Example** ($\mathcal{F}$ is rich). $u \in \mathcal{F} \cap C_c(X)$, $K = \operatorname{spt} u$, $U \supset K$, $\overline{U}$ compact. Then $\exists v \in \mathcal{F} \cap C_c(X)$, $v\big|_U = 1$.

Indeed Find open, relatively compact $V \supset \overline{U}$, $\tilde{v} := 2\varphi_{\overline{U},V} \in C_c(X)$. Pick $\tilde{\tilde{v}} \in C_c \cap \mathcal{F} : \left\|\tilde{v} - \tilde{\tilde{v}}\right\|_\infty \leqslant \varepsilon < \frac{1}{2} \Longrightarrow v := \tilde{\tilde{v}} \wedge 1 \in \mathcal{F} \cap C_c$ does the job.

Recall Riesz Representation Theorem in C_c. $I : C_c(X) \to \mathbb{R}$ positive linear functional ($\varphi \geqslant 0 \implies I\varphi \geqslant 0$). Then $\exists!$ Radon measure μ such that

$$I(\varphi) = \int \varphi \mathrm{d}\mu.$$

Radon $\mu(K) < \infty \ \forall K \subset X$ compact.

$$\mu(B) = \sup_{K\subset B,\ \text{cpt.}} \mu(K) \in [0,\infty] \qquad \text{inner regular}$$
$$= \inf_{U\supset B,\ \text{open}} \mu(U) \qquad \text{outer regular}$$

rsdf-43 **4.3 Lemma.** *$T : C_c(X) \to L^2(m)$ sub-Markovian and consider $B(f,g) := \langle Tg, f\rangle_{L^2}$ ($g, f \in C_c(X)$). Then $\exists!$ τ on $(X\times X, \mathcal{B}(X\times X))$ such that*

$$\langle Tg, f\rangle_{L^2(m)} = \int\int f(x)g(y)\tau(\mathrm{d}x,\mathrm{d}y) \tag{4.1}$$ rsdf::eq

$$\tau(N\times X) = 0 = \tau(X\times N) \text{ if } m(N) = 0 \tag{4.2}$$ rsdf::eq

$$\tau(X\times B) \leqslant m(B) \quad \forall B \in \mathcal{B}(X) \tag{4.3}$$ rsdf::eq

Note that (4.2) does *not* imply $m\otimes m$-null $\implies \tau$-null. Attention (4.3) $\tau(X\times \mathrm{d}y) \ll m(\mathrm{d}y)$

Proof. Naive argument T sub-Markovian $\overset{+\text{ cond.}}{\implies}$ T positive $\implies g \mapsto Tg(x)$ positive linear form $g \in C_c(X)$ for m-a.a. x (null sets depends on g) $\overset{\text{Riesz}}{\implies}$ $\exists!$ Radon $\mu_x(\mathrm{d}y)$: $Tg(x) = \int g(y)\mu_x(\mathrm{d}y)$

$$\implies \tau(\mathrm{d}x,\mathrm{d}y) = \mu_x(\mathrm{d}y)\ m(\mathrm{d}x) \quad \text{regular conditional probability!}$$

Proper proof $f\otimes g(x,y) := f(x)g(y)$ «Tensor product»

$$I(\varphi) = I(f\otimes g) = \langle Tg, f\rangle \text{ extends to linear map } C_c(X\times X).$$

If $I(\varphi)$ is positive, we can use Riesz for $C_c(X\times X)$. Clearly[1]

$$\varphi = \sum_{i=1}^n f_i\otimes g_i \rightsquigarrow I(\varphi) = \sum_{i=1}^n \langle Tg_i, f_i\rangle_{L^2}$$

Fix $\varepsilon > 0$.

- $f \in C_c(X) \exists$ step functions $f^\varepsilon = \sum_{i=1}^N f(x_k)\mathbb{1}_{E_k}$ with $\|f - f^\varepsilon\|_\infty \leqslant \varepsilon$[2]

[1] Fill gap: well-defined.
[2] Use f is uniformly continuous and use E_k = small box in X, E_k disjoint.

- $f_1, ..., f_n \in C_c(X) \exists$ step functions f_i^ε such that all have the same underlying partition, of course:

$$E_1 \dot\cup E_2 \dot\cup ... \dot\cup E_N = \bigcup_{i=1}^n \operatorname{spt} f_i.$$

- For $f_i, g_i \in C_c(X)$

$$\varphi = \sum_1^n f_i \otimes g_i$$
$$\varphi^\varepsilon = \sum_1^n f_i^\varepsilon \otimes g_i.$$

- Assume $\varphi \geqslant 0$, then show $I(\varphi) = \sum_1^n \langle Tg_i, f_i \rangle \overset{!!}{\geqslant} 0$

(a) We get

$$\begin{aligned} |I(\varphi) - I(\varphi^\varepsilon)| &\leqslant \sum_1^n \left| \left\langle Tg_i, f_i - f_i^\varepsilon \right\rangle_{L^2} \right| \\ &\overset{\text{sub-M}}{\leqslant} \varepsilon \sum_1^n \left\langle T\left|g_i\right|, \mathbb{1}_K \right\rangle_{L^2(m)} \xrightarrow{\varepsilon \downarrow 0} 0, \end{aligned}$$

where $\quad K = \bigcup_i^n \operatorname{spt} f_i$ compact.

(b)

$$\begin{aligned} I(\varphi^\varepsilon) &= \sum_1^n \left\langle Tg_i, f_i^\varepsilon \right\rangle_{L^2(m)} \\ &= \sum_{i=1}^n \sum_{k=1}^N \left\langle Tg_i, f_i(x_{k,i}) \mathbb{1}_{E_k} \right\rangle_{L^2(m)} \\ &= \sum_{i=1}^n \sum_{k=1}^N \left\langle T\varphi(x_{k,i}, \cdot), \mathbb{1}_{E_k} \right\rangle_{L^2(m)} \geqslant 0. \end{aligned}$$

- I is positive on span $C_c \otimes C_c$ which is dense in $C_c(X \times X)$. So I extends by density to a positive linear functional on $C_c(X \times X)$. Riesz gives: $\exists \tau(\mathrm{d}x, \mathrm{d}y)$ unique and Radon

$$I(\varphi) = \int_{X \times X} \varphi(x, y) \tau(\mathrm{d}x,\ \mathrm{d}dy).$$

- So far used only $C_c(X) \ni g \geqslant 0 \implies Tg \geqslant 0$. Now use Markov: $0 \leqslant g \leqslant \|g\|_\infty \implies 0 \leqslant Tg \leqslant \|g\|_\infty$. We know

$$\langle Tg, f \rangle = \int g(x) f(y) \tau(\mathrm{d}x, \mathrm{d}y).$$

Idea Make $g \uparrow 1$.[3]

$$\begin{aligned}\overset{\text{sub-M}}{\Longrightarrow} \langle \mathbb{1}, f\rangle_{L^2} &\geqslant \sup_{C_c^+ \ni g \geqslant 1} \langle Tg, f\rangle_{L^2} \\ &= \sup_{C_c^+ \ni g \geqslant 1} \int g \otimes f \, \mathrm{d}\tau \\ &\overset{\text{BL}}{=} \int\int \mathbb{1}_X(x) f(y) \tau(\mathrm{d}x, \mathrm{d}y) \\ &= \int_X f(y)\tau(X \times \mathrm{d}y),\end{aligned}$$

but $\langle \mathbb{1}, f\rangle_{L^2} = \int_X f(y)m(\mathrm{d}y)\ \forall f \in C_c(X)$. And it follows

$$\tau(X \times B) \leqslant m(B) \quad \forall B \in \mathcal{B}(X).$$

Indeed

$$\begin{aligned} & \int f(y)m(\mathrm{d}y) \geqslant \int f(y)\tau(X, \mathrm{d}y) && \forall f \in C_c^+(X) \\ \overset{\text{Urys.}}{\Longrightarrow} \quad & \int \mathbb{1}_K(y)m(\mathrm{d}y) \geqslant \int \mathbb{1}_K(y)\tau(X, \mathrm{d}y) && \forall K \text{ cpt.} \\ \Longleftrightarrow \quad & m(K) \geqslant \tau(X \times K) && \forall K \text{ cpt.} \\ \overset{\text{reg.}, K \subset B}{\Longrightarrow} \quad & m(B) \geqslant \tau(X \times B) && \forall B \text{ Borel}\end{aligned}$$

- $N \in \mathcal{B}(X)$, $m(N) = 0$. Then

$$0 = \langle Tg, \mathbb{1}_N\rangle_{L^2(m)} = \int\int g(x)\mathbb{1}_N(y)\tau(\mathrm{d}x, \mathrm{d}y)$$

as above $g \uparrow \mathbb{1}_X \Longrightarrow \tau(X \times N) = 0$. Assume $\hat{T}$ exists,[4] then

$$0 = \left\langle \mathbb{1}_N, \hat{T} f\right\rangle_{L^2(m)} = \int\int \mathbb{1}_N(x) f(y)\tau(\mathrm{d}x, \mathrm{d}y),$$

which is the same τ since

$$\langle Tv, u\rangle = \left\langle v, \hat{T}u\right\rangle, \quad \forall u, v \in C_c(X),$$

so both have same τ. As above $f \uparrow 1 \Longrightarrow \tau(N \times X) = 0$.

■

[3] Ok by $K_n \uparrow X$ and $K_n \subset \mathring{K}_{n+1} \subset K_{n+1}$ + Urysohn.
[4] 2 Remedies: Assume $\hat{T}$ exists and T is continuous on $L^2 \cap C_c \to L^2$.

rsdf-44 **4.4 Corollary.** *$(\mathcal{E},\mathcal{F})$ regular SDF_γ, $(\mathbf{G}_\alpha)_{\alpha>\gamma}$ resolvent and*

$$\mathcal{E}^\alpha(g,f) = \alpha\,\langle(1-\alpha\mathbf{G}_\alpha)g, f\rangle_{L^2(m)} \text{ for } g,f\in L^2(m).$$

Then $\exists!\,\sigma_\alpha(\mathrm{d}x,\mathrm{d}y)$ *such that* $\sigma_\alpha(N\times X)=0=\sigma_\alpha(X\times N)$ $(m(N)=0)$ *and such that* $\sigma_\alpha(X\times\mathrm{d}y)\ll m(\mathrm{d}y)$, $s_\alpha(y)=\frac{\sigma(X\times\mathrm{d}y)}{m(\mathrm{d}y)}\leqslant 1$ *(Radon-Nikodym density) and*

$$\begin{aligned}\mathcal{E}^\alpha(g,f) = \alpha\int(g(y)-g(x))\,f(y)\sigma_\alpha(\mathrm{d}x,\mathrm{d}y)\,+\\ \alpha\int f(x)g(x)\left(1-s_\alpha(x)\right)m(\mathrm{d}x),\end{aligned} \tag{4.4}$$ rsdf::eq04

for all $f,g\in C_c(X)$.

Proof. Use 4.2 for $T=\alpha\mathbf{G}_\alpha$ (note $\hat{T}$ exists) and $\tau=\sigma_\alpha$. This gives:

$$\langle\alpha\mathbf{G}_\alpha g,f\rangle_{L^2(m)} = \int\int g(x)f(y)\sigma_\alpha(\mathrm{d}x,\mathrm{d}y),$$

$-g(y)+g(y)+$ calculations. ■

rsdf-45 **4.5 Remark.** If $\mathcal{E}(u,v)=\mathcal{E}(v,u)=$ symmetric, then $T=\alpha\mathbf{G}_\alpha=\alpha\hat{\mathbf{G}}_\alpha=\hat{T}$ and

$$\sigma_\alpha(\mathrm{d}x,\mathrm{d}y)=\tau(\mathrm{d}x,\mathrm{d}y)=\tau(\mathrm{d}y,\mathrm{d}x)=\sigma_\alpha(\mathrm{d}y,\mathrm{d}x)$$

We swap in (4.4) $x\leftrightarrow y$, then use symmetry of σ_α.[5] Gives:

$$\begin{aligned}\frac{1}{2}\mathcal{E}^\alpha(g,f) &= \frac{1}{2}\iint A\sigma_\alpha(\mathrm{d}x,\mathrm{d}y)\\ \frac{1}{2}\mathcal{E}^\alpha(g,f) &= \frac{1}{2}\iint B\sigma_\alpha(\mathrm{d}x,\mathrm{d}y)\\ \mathcal{E}^\alpha(g,f) &= \frac{1}{2}\iint(A+B)\sigma_\alpha(\mathrm{d}x,\mathrm{d}y),\end{aligned}$$

use this to get

$$\begin{aligned}\mathcal{E}^\alpha(g,f) = \frac{1}{2}\alpha\int(g(x)-g(y))\,(f(x)-f(y))\,\sigma_\alpha(\mathrm{d}x,\mathrm{d}y)\,+\\ \alpha\int f(x)g(x)\left(1-s_\alpha(x)\right)m(\mathrm{d}x),\end{aligned} \tag{4.5}$$ rsdf::eq05

Rest of the chapter $\gamma\geqslant 0$, $\mathcal{E}(u,v)=\mathcal{E}(v,u)=$ symmetric!

5

$$\mathcal{E}^\alpha(g,f)=\alpha\int(g(x)-g(y))\,f(x)\sigma_\alpha(\mathrm{d}y,\mathrm{d}x)$$

rsdf-46 **4.6 Theorem** (Beurling-Deny, ~1960). *Let* $(\mathcal{E},\mathcal{F})$ *be regular, (symmetric)* SDF_γ, $u,v\in\mathcal{F}\cap C_c(X)$. *Then*

$$\begin{aligned}\mathcal{E}(u,v)=\mathcal{E}^{\mathrm{loc}}(u,v)&+\iint_{X\times X\setminus\mathrm{diag}}(u(x)-u(y))\,(v(x)-v(y))\,\mathcal{J}(\mathrm{d}x,\mathrm{d}y)\\&+\int_X u(x)v(x)k(\mathrm{d}x)-\gamma\int_X u(x)v(x)m(\mathrm{d}x),\end{aligned}\tag{4.6}$$

rsdf::eq

where

- $\mathcal{E}^{\mathrm{loc}}$ *is regular, strongly local (= (4.7)), symmetric bilinear form* (SDF_0?) *with* $(\mathcal{E}_1)$, $(\mathcal{E}_2)$, $(\mathcal{E}_4)$[6]

$$u,v\in\mathcal{F}\cap C_c,\ v=1 \text{ on a neighborhood of } \operatorname{spt}u\Longrightarrow\mathcal{E}^{\mathrm{loc}}(u,v)=0;\tag{4.7}$$

rsdf::eq

- $\mathcal{J}(\mathrm{d}x,\mathrm{d}y)$ *Radon on* $X\times X\setminus\mathrm{diag}$;
- $k(\mathrm{d}x)$ *Radon on* X;
- $\mathcal{J}(N\times X\setminus\mathrm{diag})=\mathcal{J}(X\times N\setminus\mathrm{diag})=0.$

Finally, $(k,\mathcal{E}^{\mathrm{loc}},\mathcal{J})$ *is uniquely determined by* $\mathcal{E}$. k *is called* **killing measure**, $\mathcal{E}^{\mathrm{loc}}$ **diffusion part**, $\mathcal{J}$ **jump measure (Lévy system)**.

Vague convergence

$$\mu_n \text{ Radon measure (unique!) } \mu_n\xrightarrow{\text{vague}}\mu\overset{\text{def}}{\Longrightarrow}\int f\,\mathrm{d}\mu_n\to\int f\,\mathrm{d}\mu\quad(f\in C_c(X))$$

For more information look at the FA refresher.

Proof. 1° **Uniqueness** Assume $\gamma=0$. Know $\mathcal{F}\cap C_c(X)$ dense in $C_c(X)$, so[7]

$$\overline{\left\{\sum_{i=1}^n u_i\otimes v_i\ :\ u_i,v_i\in\mathcal{F}\cap C_c(X),\operatorname{spt}u_i\cap\operatorname{spt}v_i=\varnothing\right\}}=C_c(X\times X\setminus\mathrm{diag})$$

by (4.6) we get for $u,v\in\mathcal{F}\cap C_c(X)$, $\operatorname{spt}u\cap\operatorname{spt}v=\varnothing$.

$$\mathcal{E}(u,v)=-2\iint_{X\times X\setminus\mathrm{diag}}u(x)v(y)\mathcal{J}(\mathrm{d}x,\mathrm{d}y)$$

$\overset{\text{density of span}}{\Longrightarrow}$ $\mathcal{J}$ uniquely determined by $\mathcal{E}$.

[6] Idea: $\int\nabla u\nabla u=\int_{\operatorname{spt}u}\nabla u\underbrace{\nabla 1}_{=0}=0.$

[7] Note that, C is compact in $X\times X\setminus\mathrm{diag}\iff C$ is compact in $X\times X$ and C does not meet an ε-neighborhood of diag ($\varepsilon=\varepsilon(C)$).

Take $u, v \in C_c(X) \cap \mathcal{F}$, $v = 1$ in a neighborhood of $\operatorname{spt} u$ (see Example 4.2). By (4.6) and strong local ($\mathcal{E}^{\mathrm{loc}} = 0$), $\gamma = 0$

$$\mathcal{E}(u, v) - \iint_{X \times X \setminus \mathrm{diag}} (u(x) - u(y))\,(v(x) - v(y))\,\mathcal{J}(\mathrm{d}x, \mathrm{d}y) = \int_X u(x) k(\mathrm{d}x)$$

$\Longrightarrow$ k uniquely determined by $\mathcal{E} + \mathcal{J}$, i.e. by $\mathcal{E}$.

$\Longrightarrow$ also $\mathcal{E}^{\mathrm{loc}}$ unique.

2^{o} **Construction of $\mathcal{J}(\mathrm{d}x, \mathrm{d}y)$** . Use $\Longrightarrow \sup_{\alpha > \gamma} \left| \iint u \otimes v \alpha \mathrm{d}\sigma_\alpha \right| < \infty$

$\Longrightarrow \sup_{\alpha > \gamma} \alpha \sigma_\alpha (K \times K \setminus \{(x, y) : d(x, y) < \varepsilon\} < \infty\ \forall K \subset X$ cpt.

$(\alpha \sigma_\alpha)\alpha > \gamma$ is vaguely bounded on $X \times X \setminus \mathrm{diag}$

$\underset{\text{fa-refresher}}{\overset{\text{cpt.ness}}{\Longrightarrow}} \exists \alpha_n : \alpha_n \sigma_{\alpha_n} \xrightarrow{\text{vaguely}} 2\mathcal{J}$, i.e.

$$\lim_n \iint \varphi(x, y) \alpha_n \sigma_{\alpha_n}(\mathrm{d}x, \mathrm{d}y) = 2 \iint \varphi(x, y) \mathcal{J}(\mathrm{d}x, \mathrm{d}y) \qquad (\forall \varphi \in C_c(X \times X \setminus \mathrm{diag}))$$

In particular: $\varphi = u \otimes v$ and $\operatorname{spt} u \cap \operatorname{spt} v = \emptyset$. Hence,

$$\mathcal{E}(u, v) = \lim_n \mathcal{E}^{\alpha_n}(u, v) = -2 \iint u \otimes v \mathrm{d}\mathcal{J}.$$

3^{o} **Construction of k** U_i open, $\overline{U}_i$ compact, $U_i \uparrow X$, $\delta_i > 0$, $\delta_i \downarrow 0$. Set

$$\Gamma_i = U_i \times U_i \setminus \left\{(x, y) : d(x, y) < \delta_i\right\}.$$

WLOG $\mathcal{J}(\partial \Gamma_i) = 0$[8]. Since $u \in C_c(X) \cap \mathcal{F}$, $\operatorname{spt} u \subset U_i$ we get

$$\mathcal{E}^\alpha(u, v) = \frac{1}{2} \alpha \iint_{U_i \times U_i} (u(x) - u(y))^2\, \sigma_\alpha(\mathrm{d}x, \mathrm{d}y) + \alpha \int_{U_i} u^2(x)(1 - s_\alpha(x)) m(\mathrm{d}x),$$

with $\sigma_\alpha = \frac{\mathrm{d}\sigma_\alpha(X \times (U_i \cap \cdot))}{\mathrm{d}m\big|_{U_i}}$. Then

$$\left(\left(1 - s_\alpha(x)\right) m(\cdot \cap U_i)\right)_{\alpha > \gamma} \text{ vaguely bounded.}$$

So, $\exists \alpha_n \uparrow \infty : (1 - s_{\alpha_n}(x)) m(\cdot \cap U_i) \xrightarrow{\text{vaguely}} k_i$ (in U_i). Define k by gluing together:

$$\int \varphi \mathrm{d}k = \int_{U_i} \varphi \mathrm{d}k_i \quad \text{if } \varphi \in C_c(U_i).$$

[8] $\overline{\Gamma}_i$ compact, by, construction, and $\mathcal{J}$ is Radon, i.e. finite on compacts. Use Cavalieri. Exercise: m measure on $[0, 1]$, $m[0, 1] < \infty$. Then $\exists$ at most countably many atoms in $[0, 1]$ and $\exists$ many set $(a, b) \subset [0, 1]$ with $m\{a\} = m\{b\} = 0$.

Hence,

$$\begin{aligned}\mathcal{E}(u,u) &= \lim_n \frac{1}{2}\alpha_n \iint_{U_i\times U_i\cap\{d(x,y)<\delta_i\}} (u(x)-u(y))^2\,\sigma_{\alpha_n}(\mathrm{d}x,\mathrm{d}y)\\ &\quad+\iint_{\Gamma_i}(u(x)-u(y))^2\,\mathfrak{J}(\mathrm{d}x,\mathrm{d}y)\\ &\quad+\int_{U_i} u^2(x)k(\mathrm{d}x),\end{aligned}$$

where $\operatorname{spt} u \subset U_i$. Letting $i\to\infty$ is no problem in last 2 integrals (Beppo Levi), so

$$\lim_i \lim_n \frac{1}{2}\alpha_n \iint_{U_i\times U_i\cap\{d(x,y)<\delta_i\}} (u(x)-u(y))^2\,\sigma_{\alpha_n}(\mathrm{d}x,\mathrm{d}y)$$

exists and defines $\mathcal{E}^{\mathrm{loc}}(u,v)$ by polarization. i.e.

$$\mathcal{E}^{\mathrm{loc}}(u,v) = \frac{1}{2}\left(\mathcal{E}^{\mathrm{loc}}(u+v,u+v)-\mathcal{E}^{\mathrm{loc}}(u)-\mathcal{E}^{\mathrm{loc}}(v)\right).$$

Clear $(\mathcal{E}_1),(\mathcal{E}_2),(\mathcal{E}_4)$ as integral has $(\mathcal{E}_4)$ and the limit preserves it.

■

Q+R

1° Are components of (4.6) SDF$_\gamma$? $(\mathcal{E}_3)$ for $\mathcal{E}^{\mathrm{loc}}$?

2° $\mathcal{E}^{\mathrm{loc}}$ is often a differential expression, see §5: $\mathcal{E}^{\mathrm{loc}} \approx \int \nabla u\,\nabla v$. Plug in $\frac{(u(x)-u(y))^2}{|x-y|^2}\ |x-y|^2$ in the integral. Then $|\nabla u|^2$ as $|x-y|\to 0$. Stochastic interpretation: We get a diffusion from a jump process.

3° (4.6) seems to hold (in this generality) only on $\mathcal{F}\cap C_c(X)$, not on $\mathcal{F}$. Different from (4.4) and (4.5). (4.4) and (4.5) extend from $C_c(X)$ to $L^2(m)$ since $\alpha\sigma_\alpha(X\times N) = \alpha\sigma_\alpha(N\times N)=0$ (vague limit) if $m(N)=0$.

Reason $g(x) := \frac{1}{\sqrt{2\pi}}$, $g_\varepsilon(x) := g\left(\frac{x}{\varepsilon}\right)\frac{1}{\varepsilon}$, then $g_\varepsilon(x)\mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{\text{vague}} \delta_0$.

Chapter 5

Examples

Crash course on Sobolev spaces. $D \subset \mathbb{R}^n$ open.

ex-51 **5.1 Definition.** (a) $u \in L^p_{\text{loc}} \iff \forall \chi \in C_c^\infty \,:\, \chi \cdot u \in L^p$

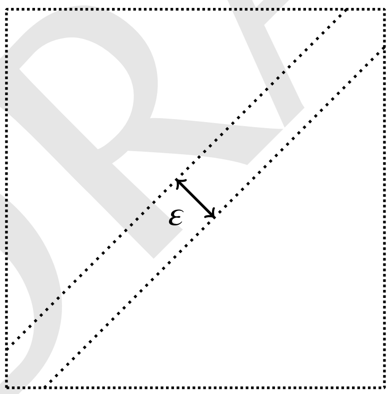
(b) $u \in L^1_{\text{loc}}$ **has a weak (distributional) derivative** if $\exists v \in L^1_{\text{loc}}$ such that

$$\int_D v\,\varphi\,\mathrm{d}x = -\int_D u\,\partial_j\varphi\,\mathrm{d}x, \qquad (\forall \varphi \in C_c^\infty(D))$$

then $v = \partial_j u$.

(c) The L^2-**Sobolev space** $W^k(D)$ $(k \in \mathbb{N})$ is

$$W^k(D) := \left\{ u \in L^1_{\text{loc}}(D) \,:\, u \in L^2(D),\ \partial^\alpha u \in L^2(D),\ |\alpha| \leqslant k,\ \alpha \in \mathbb{N}_0^n \right\}$$



Need Friedrichs mollifier

(1) $j \in C_c^\infty(\mathbb{R}^n)$, $0 \leqslant j \leqslant 1$, $\operatorname{spt} j \subset \overline{B_1(0)}$, $\int j(x)\mathrm{d}x = 1$.

(2) $j_h(x) := h^{-n} j\left(\frac{x}{h}\right)$, $0 \leqslant j \leqslant 1$, $\operatorname{spt} j_h \subset \overline{B_h(0)}$, $\int j_h(x)\mathrm{d}x = 1$.

(3) $u \in L^p_{\text{loc}}(\mathrm{d}x) = L^p_{\text{loc}}(\mathrm{d}x) \cup C \,:\, u_h(x) := j_h * u(x) = h^{-n} \int j\left(\frac{x-y}{h}\right) u(y)\mathrm{d}y \in C^\infty$.

ex-52 **5.2 Lemma.** $u \in C(D) \Longrightarrow u_h \xrightarrow[h\downarrow 0]{\text{locally uniformly}} u$.

Proof. $D' \subset\subset D$, i.e. D' open, $\overline{D'} \subset D$ compact, and assume $\operatorname{dist}(D', \partial D) > h$.

$$u_h(x) = h^{-n} \int_{|x-y| \leqslant h} \underbrace{j\left(\frac{x-y}{h}\right)}_{=:\, z} u(y) \mathrm{d}y$$
$$= \int_{|z| \leqslant 1} j(z)\, u(x - hz) \mathrm{d}z$$

and

$$\left| u(x) - u_h(x) \right| \leqslant \int_{|z|\leqslant 1} j(z)\, |u(x) - u(x - hz)|\, \mathrm{d}z$$
$$\leqslant \sup_{x \in D'} \sup_{|z| \leqslant 1} |u(x) - u(x - hz)|$$
$$\xrightarrow[\text{uniform cts}]{h \to 0} 0.$$

Mind: $D' + \overline{B_h(0)} \subset\subset D$. ■

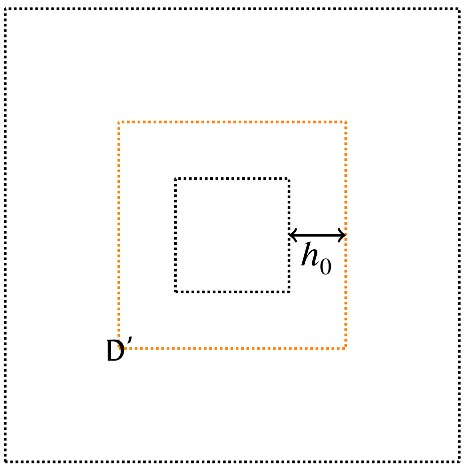
ex-53 **5.3 Remark.** If $u \in C(\overline{D})$ and $u\big|_{\partial D} = 0$ (if D unbounded: u vanishes at ∞, at least in certain directions; e.g. if $D = \mathbb{R}^n$ u vanishes at ∞), then convergence is uniform on D.

ex-54 **5.4 Corollary.** *$u \in L^p_{\mathrm{loc}}$, $p < \infty$, then $u \xrightarrow[L^p_{\mathrm{loc}}]{h \downarrow 0} u$.*

Proof. WLOG D bounded. Otherwise consider $u\chi$ with $\chi \in C_c^\infty$ and $\operatorname{spt} \chi + B_1(0) =: D$. By Jensen's inequality, we get

$$\left|u_h(x)\right|^p \leqslant \int_{|z|\leqslant 1} j(z)\, |u(x - hz)|^p\, \mathrm{d}z$$

Fix $h_0 > h > 0$, pick $D' \subset\subset D'$ and assume WLOG $D' \subset\subset D' + B_{3h}(0) \subset\subset D$.



$$
\begin{aligned}
\int_{D'} |u_n|^p \,\mathrm{d}x &\leqslant \int_{D'} \int_{|z|\leqslant 1} j(z)\,|u(x-hz)|^p \,\mathrm{d}z\mathrm{d}x \\
&\overset{\text{Tonelli}}{=} \underbrace{\int_{|z|\leqslant 1} j(z)\mathrm{d}z}_{=1} \underbrace{\int_{D'} |u(x-hz)|^p \,\mathrm{d}x}_{\leqslant \int_{D'+B_{h_0}(0)} |u(y)|^p \mathrm{d}y} \\
&\leqslant \int_{D'+B_{h_0}(0)} |u|^p \,\mathrm{d}x \forall h < h_0
\end{aligned}
$$

$\Longrightarrow \exists w \in C_c(D) : \|u-w\|_{L^p(D'+B_{h_0}(0))} \leqslant \varepsilon$. Then Lemma 5.2), $\mathrm{Leb}(D) < \infty$:

$$
\exists h < h_0 : \left\|w - w_h\right\|_{L^p(D'+B_{h_0}(0)} \leqslant \varepsilon.
$$

Now 3ε trick:

$$
\left\|u-u_n\right\|_{L^p(D'+B_{h_0}(0))} \leqslant \underbrace{\|u-w\|}_{\leqslant \varepsilon} + \underbrace{\left\|u-w_h\right\|}_{\leqslant \varepsilon} + \underbrace{\left\|w_h - u_n\right\|}_{=\left\|((w-u)_h\right\| \leqslant \|w-u\| \leqslant \varepsilon} \leqslant 3\varepsilon
$$

■

ex-55 **5.5 Lemma.** *$u \in L^1_{\mathrm{loc}}(D)$, $\partial_i u$ exists (weakly), $h < \mathrm{dist}(x_0, \partial D)$. Then $\partial_i(u_h)(x_0) = (\partial_i u)_h(x_0)$.*

Proof. Let $j_h \in C_c^\infty$, as the weak derivative exists (and ok, it is under the integral sign)

$$
\begin{aligned}
\partial_i(u_h) &= \partial_i(j_h * u) \\
&= (\partial_i j_h) * u \\
&= j_h * (\partial_i u) \\
&\overset{\text{def}}{=} (\partial_i u)_h
\end{aligned}
$$

■

ex-56 **5.6 Theorem.** *$u, v \in L^1_{\mathrm{loc}}(D)$. Then*

$$
v = \partial_i u \Longleftrightarrow \exists (u_n)_n \subset C^\infty(D) : u_n \xrightarrow{L^1_{\mathrm{loc}}} u,\ \partial_i u_n \xrightarrow{L^1_{\mathrm{loc}}} v.
$$

Proof. $\Longrightarrow$ Lemma 5.4 and 5.5, take $h = \frac{1}{n}$.

$\Longleftarrow$ Definition of weak derivative (you «test» w.r.t $\varphi \in C_c^\infty(D)$. ■

ex-57 **5.7 Lemma** (Chain rule). *Let $\varphi \in C_b^1(\mathbb{R})$, $u \in W^1(\mathbb{R})$. Then $\varphi \circ u \in W^1(D)$, $\nabla(\varphi \circ u) = \varphi'(u)\nabla u$.*

Proof. By 5.6 there exists $(u_n) \subset C^\infty(D)$ such that

$$u_n \xrightarrow{L^1_{\text{loc}}} u, \quad \nabla u_n \xrightarrow{L^1_{\text{loc}}} \nabla u$$

for a *subsequence* we may assume a.e. convergence. But depends on $D' \subset\subset D$

$$\chi\, u_n \xrightarrow{L^1_{\text{loc}}} \chi\, u, \quad \nabla \chi\, u_n \xrightarrow{L^1_{\text{loc}}} \chi\, \nabla u \forall \chi \in C_c(D).$$

WLOG, no change in name for the subsequence. Let $U \subset\subset D$, then

- $$\int_U \left|\varphi(u_n) - \varphi(u)\right| \mathrm{d}x \underset{\text{theorem}}{\overset{\text{mean value}}{\leqslant}} \|\varphi'\|_\infty \int_U \left|u - u_n\right| \mathrm{d}x \xrightarrow{n\uparrow 0} 0$$
- $$\int_U \left|\varphi'(u_n)\nabla u_n - \varphi'(u)\nabla u\right| \mathrm{d}x$$
$$\leqslant \underbrace{\int_U \|\varphi'\|_\infty \left\|\nabla u_n - \nabla u_n\right\| \mathrm{d}x}_{\xrightarrow{n\uparrow\infty} 0} + \underbrace{\int_U \underbrace{\left|\varphi'(u_n) - \varphi'(u)\right|}_{\xrightarrow{\text{a.s.}} 0} \underbrace{\left|\nabla u\right|}_{\in L^1(u)} \mathrm{d}x}_{\to 0 \text{ (DOM)}}$$

So $\forall v \in C_c^\infty(D)$:

$$\int \nabla\varphi(u_n)\, v\mathrm{d}x \overset{\text{parts}}{=} -\int \varphi(u_n)\, \nabla v\mathrm{d}x$$
$$\int \varphi'(u_n)\, \nabla u_n\, v\mathrm{d}x \overset{\text{parts}}{=} -\int \varphi(u_n)\, \nabla v\mathrm{d}x$$

This goes to

$$\int \varphi'(u)\, \nabla u\, v\mathrm{d}x = -\int \varphi(u)\, \nabla u\mathrm{d}x,$$

by definition we see $\nabla\varphi(u) = \varphi'(u)\nabla u$.

■

ex-58 **5.8 Corollary.** *$u \in W^1(D) \implies u^+ \in W^1(D)$ and $\nabla u^+ = \mathbb{1}_{\{u>0\}}\nabla u$. In particular, $u \wedge b \in W^1(D)\ \forall b \in \mathbb{R}$.*

Proof. Idea $u^+ = \varphi \circ u$, $\varphi(x) = x \vee 0$.
Smooth out:

$$\varphi_\varepsilon(u) = \begin{cases} \sqrt{u^2 + \varepsilon^2} - \varepsilon & , u > 0 \\ 0 & , u \leqslant 0 \end{cases}$$

is differentiable. By Lemma 5.7, for all $v \in C_c^\infty(D)$

$$\int \varphi_\varepsilon(u)\ \nabla v \mathrm{d}x = -\int_{u>0} \frac{u}{\sqrt{u^2+\varepsilon^2}} \nabla u\ v \mathrm{d}x$$

$$\varepsilon \to 0 \qquad \int u^+\ \nabla v \mathrm{d}x = -\int_{u>0} \nabla u\ v \mathrm{d}x.$$

Hence by definition ∇u^+ exists and $\nabla u^+ = \mathbb{1}_{\{u>0\}} \nabla u$ (weak derivative!). Finally, $u \wedge b = u - (u-b)^+$. ■

Now we investigate $W^k(D)$ as Hilbert spaces.

$$\langle u, v \rangle_{W^k(D)} := \sum_{0 \leqslant |\alpha| \leqslant k} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(D, \mathrm{d}x)},$$

$\partial^0 u \overset{\text{def}}{=} u$, captures L^2-behaviour of all derivatives, with norm

$$\|\cdot\|_{W^k(D)} := \sqrt{\langle \cdot, \cdot \rangle_{W^k(D)}}.$$

Subordinate partition of unity $U_i \subset\subset D$, $\bigcup_{i \in \mathbb{N}} U_i = D$. Then $\exists \psi_i \in C_c^\infty(U_i)$ and $\sum_i \psi_i \big|_D = 1$.

ex-59 **5.9 Theorem.** $\overline{C^\infty(D) \cap W^k(D)}^{\|\cdot\|_{W^k(D)}} = W^k(D)$.

Attention

- $C^\infty(\overline{D}) \cap W^k(D)$ in general *not* dense (smoothness/geometry of ∂D),
- $C_c^\infty(D) \cap W^k(D) \overset{\text{Excerise}}{=} C_c^\infty(D)$ in general not dense (reason: all is $= 0$ on ∂D).

Hence, we define

$$W_0^k(D) := \overline{C_c^\infty(D)}^{\|\cdot\|_{W^k(D)}} \subsetneq W^k(D).$$

In particular, $W_0^k(\mathbb{R}^n) = W^k(\mathbb{R}^n)$.

Proof of 5.9. Pick $V_1 \subset\subset V_2 \subset\subset V_3 \subset\subset \ldots \uparrow D$ and $U_i := V_{i+1} \setminus \overline{V_{i-1}}$, $V_0 = V_{-1} := \emptyset$. $(\psi_i)_i$ is the corresponding partition of unity. Fix $u \in W^k(D)$, $\varepsilon > 0$. Now use Corollary 5.4 to get

$$\forall i \in \mathbb{N}\ \exists h_i > 0,\ h_i < \operatorname{dist}(V_{i+1}, \partial D) : \sum_{|\alpha| \leqslant k} \left\| \partial^\alpha (\psi_i u)_{h_i} - \partial^\alpha (\psi_i u) \right\|_{L^2(D)} \leqslant \frac{\varepsilon}{2^i}. \tag{5.1}$$ ex::eq04

For all $D' \subset\subset D$, (5.1) guarantees that only finitely many $\partial^\alpha(\psi_i u) \neq 0$ on D'[1] Then,

$$\partial^\alpha v = \sum_{\text{finitely locally}} \partial^\alpha(\psi_i u)_{h_i} \in C^\infty(D),$$

and, by (5.1) and $\sum \psi_i = 1$, triangle inequality,

$$\sum_{|\alpha|\leqslant k} \|\partial^\alpha u - \partial^\alpha v\|_{L^2(D)} \leqslant \sum_{|\alpha|\leqslant k} \sum_{i\in\mathbb{N}} \left\|\partial^\alpha(\psi_i u)_{h_i} - \partial^\alpha(\psi_i u)\right\|_{L^2}$$
$$\overset{(5.1)}{\leqslant} \sum_{i\in\mathbb{N}} \frac{\varepsilon}{2^i} = \varepsilon.$$

■

Assume now $D = \mathbb{R}^n$.

ex-510 **5.10 Corollary.** $\overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{W^k(\mathbb{R}^n)}} = W^k(\mathbb{R}^n)$.

Proof. $C_c^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, $\forall u \in C_c^\infty(\mathbb{R}^n) : \partial^\alpha u \in C_c^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \Longrightarrow C_c^\infty(\mathbb{R}^n) \subset W^k(\mathbb{R}^n)$.

Know $C^\infty(\mathbb{R}^n) \cap W^k(\mathbb{R}^n)$ is dense in $W^k(\mathbb{R}^n)$.

So it is enough to prove

$$\forall \varepsilon > 0\ \forall u \in C^\infty(\mathbb{R}^n) \cap W^k(\mathbb{R}^n)\ \exists \varphi \in C_c^\infty(\mathbb{R}^n) : \|u - \varphi\|_{W^k(D)} \leqslant \varepsilon$$

Fix $\varepsilon > 0$, $u \in C^\infty(\mathbb{R}^n) \cap W^k(\mathbb{R}^n)$. Pick $\chi \in C_c^\infty(\mathbb{R}^n)$, $0 \leqslant \chi \leqslant 1$, $\chi\big|_{B_1(0)} = 1$, $\chi\big|_{B_2^c(0)} = 0$ and

$$u_i(x) := u(x)\chi\left(\frac{x}{i}\right) \in C_c^\infty(\mathbb{R}^n).$$

So $u_i\big|_{B_i(0)} = u\big|_{B_i(0)}$ and

$$\begin{aligned} \left|\partial^\alpha u_i\right| &= \left|\partial^\alpha u \chi\left(\frac{\cdot}{i}\right)\ \right| \\ &= \left|\sum_{\alpha=\beta+\gamma} \binom{\alpha}{\gamma} \partial^\beta u\ \partial^\gamma \chi\left(\frac{\cdot}{i}\right)\right| \\ &\leqslant c_{\alpha,\chi} \sum_{|\beta|\leqslant k} \left|\partial^\beta u\right| \quad \forall \alpha \in \mathbb{N}_0,\ |\alpha| \leqslant k. \end{aligned}$$

[1]By compactness: $\overline{D}$ meets only finitely many of the U_i.

Thus,[2]

$$\underbrace{\sum_{|\alpha|\leqslant k}\int \left|\partial^\alpha(u-u_i)\right|^2 \mathrm{d}x}_{=\|u-u_i\|_{W^k(D)}} \leqslant c' \sum_{|\alpha|\leqslant k}\int_{|x|\geqslant i} |\partial^\alpha u|^2 \,\mathrm{d}x \xrightarrow{i\uparrow\infty} 0. \qquad \blacksquare$$

ex-11 **5.11 Example.** Classical Dirichlet form SDF_0, i.e. symmetric, $\gamma = 0$ and $D = \mathbb{R}^n$, $m(\mathrm{d}x) = \mathrm{d}x$. Then

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^n} \nabla u(x)\cdot\nabla v(x)\mathrm{d}x \qquad (5.2) \quad \text{ex::eq05}$$

$$= \frac{1}{2}\langle \nabla u, \nabla v\rangle_{L^2(\mathrm{d}x)} \qquad (5.3) \quad \text{ex::eq06}$$

(5.2) and (5.3) is ok for $u, v \in C_c^\infty(\mathbb{R}^n)$ and $u, v \in W^1(\mathbb{R}^n)$. Moreover,

- $\mathcal{E}(u,u) \geqslant 0 \longrightarrow (\mathcal{E}_1), \gamma = 0$
- $|\mathcal{E}(u,v)| \leqslant \sqrt{\mathcal{E}(u)}\sqrt{\mathcal{E}(v)} \longrightarrow (\mathcal{E}_2)$ (b/o symmetry)
- $\mathcal{E}(u,v) = \mathcal{E}(v,u) \longrightarrow$ symmetry
- $\mathcal{F} = W^1(\mathbb{R}^n)$, $(\mathcal{E},\mathcal{F})$ closed, cf. Example **??** or $(W^1, \mathcal{E}_\alpha)_{\alpha>0}$ Hilbert
- Contraction: Corollary 5.8, $u \in W^1(\mathbb{R}^n) \Longrightarrow u \wedge b \in W^1(\mathbb{R}^n)$ and

  $$\nabla(u\wedge b) = \nabla\left(u - (u-b)^+\right) = \nabla u - \nabla(u-b)^+$$
  $$= \nabla u - \nabla u \mathbb{1}_{\{u>b\}} = \mathbb{1}_{\{u\leqslant b\}}\nabla u$$

  Hence,

  $$\mathcal{E}(u\wedge b, u) = \int_{\mathbb{R}^n} \nabla(u\wedge b)\nabla u \mathrm{d}x = \int_{u\leqslant b} |\nabla u|^2 \,\mathrm{d}x \begin{cases} = \mathcal{E}(u\wedge b) & \longrightarrow (\mathcal{E}_4) \\ \leqslant \mathcal{E}(u) & \text{«normal contraction prop.»} \Longleftarrow \end{cases}$$

- Regularity: Corollary 5.10
  (a) $C_c^\infty(\mathbb{R}^n) \cap W^1(\mathbb{R}^n)$ dense in $W^1(\mathbb{R}^n)$
  (b) $C_c^\infty(\mathbb{R}^n) \cap W^1(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$ so dense, too.

ex-512 **5.12 Example.** $D \subset \mathbb{R}^n$ bounded domain, open (and connected), ∂D is C^1-curve. Define[3]

$$\mathcal{E}(u,v) = \frac{1}{2}\int_D \nabla u\ \nabla v \mathrm{d}x \qquad (\forall u, v \in C_c^\infty(D))$$

$$\mathcal{F} := W_0^1(D) := \left\{u \in W^1(D) : u\big|_{\partial D} = 0\right\} \text{ gives regularity.}$$

[2]Recall: $|a-b|^2 \leqslant 2\left(a^2+b^2\right)$.
[3]Note that there is no problem at ∂D, as $\operatorname{spt} u \subset\subset D$.

As in ?? we get $(\mathcal{E}_1) - (\mathcal{E}_4)$, even regular, since (not obvious) $\overline{C_c^\infty(D)}^{\|\cdot\|_{W^1(D)}} = W_0^1(D)$.

ex-513 **5.13 Scholium** (On traces, [Trial]). Let D be bounded, ∂D a C^1-curve. Then there exists a good $(n-1)$–dimensional surface measure on ∂D (Hausdorff measure).

Problem with $u\big|_{\partial D}$ ∂D is $(n-1)$-dimensional, smooth $\Longrightarrow \mathrm{Leb}_{\mathbb{R}^n}(\partial D) = 0$

$u \in W^1(D) \subset L^2(D, \mathrm{d}x)$, $u(x) = ?\ \forall x \in \partial D)$. u does not see null sets, i.e. $u\big|_{\partial D}$ = no good sense *naively*. Way out:

1° $u \in \overline{C}^\infty(D) = \Big\{u \in C^\infty(D) : \partial^\alpha u \in C(\overline{D}),$ if D not bounded $\operatorname{spt} u \subset B_R(0) \cap D, R = R_u\Big\}$
$u\big|_{\partial D}$ makes sense (pointwise defined) and

- $\|u\|_{L^2(D,\mathrm{d}x)} \leqslant \|u\|_{W^1(D)}$
- $\|u\|_{L^2(\partial D,\mathrm{d}S(\partial D))} \leqslant \|u\|_{W^1(D)}$ (not easy[4])

2° $u \in W^1(D)\ \exists (u_n)_n \subset \overline{C}^\infty(D) \cap W^1(D)$ with $u_n \xrightarrow{W^1(D)} u \overset{1^\circ}{\Longrightarrow} (u_n)$ Cauchy in $L^2(\partial D)$ and hence

$$u\big|_{\partial D} := \lim_{n\to\infty} u_n \text{ in } L^2(\partial D)$$

the so called **Trace**. The Trace depends on the smoothness of ∂D and the weak order of differentiability.

3° Take $\varphi \in \overline{C}^1(D) = \Big\{\varphi \in C^1(D), \partial^\alpha \varphi \in C(\overline{D}), |\alpha| = 0, 1\Big\}$ and $u \in W^1(D), \overline{C}^\infty(D) \cap$ $W^1(D) \ni u_j \xrightarrow{W^1(D)} u$ as in 2°. Then[5]

$$\int_D \frac{\partial}{\partial x_j} u_j \cdot \varphi \mathrm{d}x \overset{\text{Gauß}}{=} \int_{\partial D} u_j\ \varphi\ \cos(\nu x_i) \mathrm{d}S(x) - \int_D u_j \frac{\partial}{\partial x_j} \varphi \mathrm{d}x$$

$$j \uparrow \infty \qquad \int \frac{\partial}{\partial x_i} u\ \varphi \mathrm{d}x = \int_{\partial D} u\big|_{\partial D}\ \varphi\ \cos(\nu x_i) \mathrm{d}S(x) - \int_D u \frac{\partial}{\partial x_i} \varphi \mathrm{d}x$$

So in $W_0^1(D)$ we have $u\big|_{\partial D} = 0$ and integration by parts simplifies.

ex-514 **5.14 Definition.** Let D be a bounded C^1-domain. Then

$$W_0^1(D) := \left\{u \in W^1(D) : u\big|_{\partial D} = 0\right\}.$$

Useful facts

[4]Note that, we get even $W^\alpha(D)$ for $\alpha > \frac{1}{2}$, so D controls the boundary
[5]νx_i a bite sloppy for die angle of outer normal ν at x with e_i, $\mathrm{d}S$ measure on ∂D.

(1) $W^1_0(D)$ *closed* subspace of $W^1(D)$. Hence, Hilbert with same norm as $W^1(D)$

(2) $\overline{C_c^\infty(D)}^{\|\cdot\|_{W^1(D)}} = W^1_0(D)$

(3) $\underbrace{W^1_0(D)}_{u|_{\partial D}=0 \Longrightarrow 1\notin W^1_0(D)} \subsetneq W^1(D)$ and $W^1(D)$ contains $u = 1$, $\nabla 1 = 0, 1\big|_{\partial D} = 0$.

ex-515 **5.15 Remark.** $\mathcal{E}(u, v) = \frac{1}{2}\int_D \nabla u\ \nabla v \mathrm{d}x$.

D	$\mathcal{F}$	comment
$\mathbb{R}^n$	$W^1(\mathbb{R}^n) = W^1_0(\mathbb{R}^n)$	$(\mathcal{E}_1) - (\mathcal{E}_4)$ regular
open, bdd, C^1-∂D	$W^1_0(D)$	$(\mathcal{E}_1) - (\mathcal{E}_4)$ regular, leads to Dirichlet problem
open, bdd, C^1-∂D	$W^1(D)$	$(\mathcal{E}_1) - (\mathcal{E}_4)$ *not* regular, leads to Neumann problem

Aim Identify generator $\mathbf{A}$, semigroup T_t[6] for $\mathcal{E}$

Recall from 3.5

$$\mathcal{E}^\alpha(u, v) = \alpha\left\langle(1 - \alpha\mathbf{G}_\alpha)u, v\right\rangle_{L^2(\mathrm{d}x)} \xrightarrow{u,v\in\mathcal{F}} \mathcal{E}(u, v)$$

gives relation $\mathbf{G}_\alpha$, $\mathbf{A}$ and $\mathcal{E}$.

Recall from Hille-Yosida (2.7, 2.10)

$$\alpha(1 - \alpha\mathbf{G}_\alpha) = \alpha\left(1 - \frac{\alpha}{\alpha - \mathbf{A}}\right) = \alpha\frac{\alpha - \mathbf{A} - \alpha}{\alpha - \mathbf{A}} = -\alpha\mathbf{G}_\alpha \overset{\text{on } \mathfrak{D}(\mathbf{A})}{=} -\alpha\mathbf{G}_\alpha\mathbf{A} \xrightarrow[\text{by 2.7}]{\alpha\uparrow\infty} -\mathbf{A} \text{ on } \mathfrak{D}(\mathbf{A})$$

Hence,

$$\mathcal{E}(u, v) = \langle -\mathbf{A}u, v\rangle_{L^2(m)} \begin{cases} u, v \in \mathcal{F} \\ u \in \mathfrak{D}(\mathbf{A}),\ v \in L^2(m) \end{cases}$$

we can find $-\mathbf{A}$ by «integration by parts».

ex-516 **5.16 Example.**

$$\mathcal{E}(u, v) = \frac{1}{2}\int_D \nabla u\ \nabla v,\ \mathcal{F} = W^1_0(D)$$

D bounded, C^1-∂D (or $D = \mathbb{R}^n$). By Gauß' theorem:

$$\mathcal{E}(u, v) = -\frac{1}{2}\int_D \Delta u\ v \mathrm{d}x,$$

[6] Only if $D = \mathbb{R}^n$, see later.

where $u \in W_0^2(D)$, i.e.

$$u\big|_{\partial D} = \frac{\partial}{\partial x_i} u\big|_{\partial D} = 0 \ \forall i = 1, \dots, n,$$

and $v \in W_0^1(D)$

Show $\left(-\frac{1}{2}\Delta, W_0^2(D)\right)$ is a closed operator, so it is generator of $(\mathbf{G}_\alpha)_{\alpha>0}$ and of $\mathcal{E}$. Now find the semigroup. Consider the following initial value on $D = \mathbb{R}^n$

$$\begin{aligned} \partial_t w(x,t) &= \frac{1}{2}\Delta_x w(x,t) & x \in \mathbb{R}^n, t > 0 \\ w(x,0) &= f(x) & f \in L^2(D) \end{aligned}$$

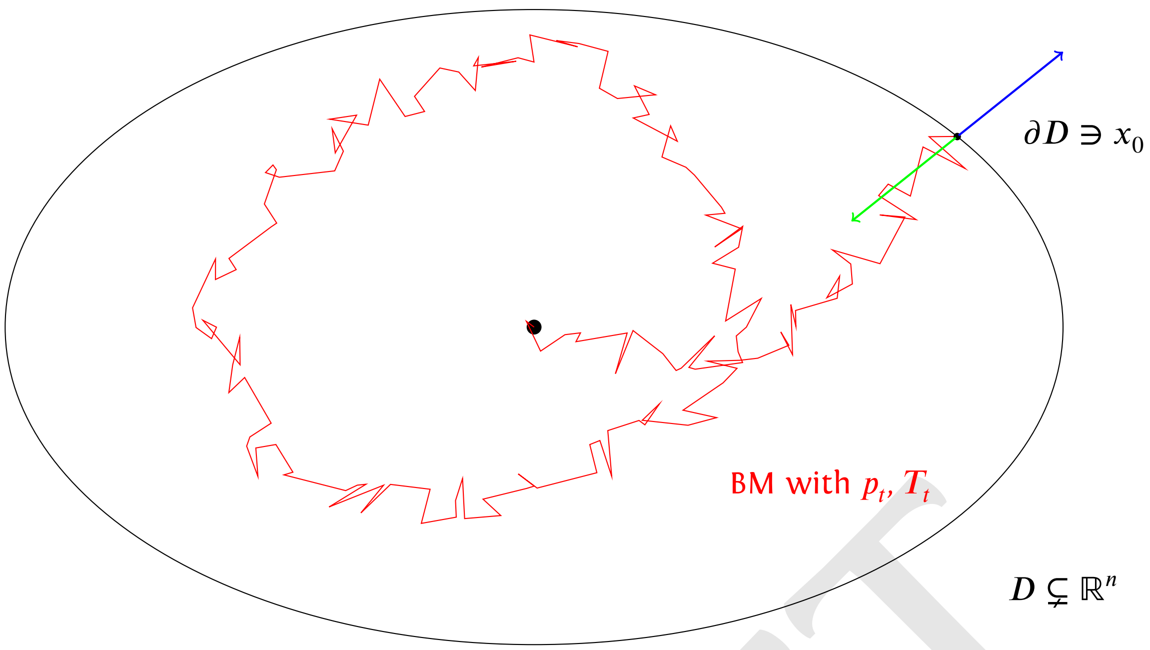
So we can use Fourier methods:

$$\begin{aligned} \partial_t \hat{w}(\xi,t) &= \hat{\Delta_x w}(\xi,t) \\ \hat{w}(\xi,0) &= \hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} \mathrm{d}x \end{aligned}$$

This is an ODE in t, hence,

$$\begin{aligned} \Longrightarrow \qquad \frac{\partial_t \hat{w}(\xi,t)}{\hat{w}(\xi,t)} &= -\frac{1}{2}|\xi|^2 \\ \Longrightarrow \qquad \hat{w}(\xi,t) &= \hat{f}(\xi) e^{-\frac{1}{2}t\ |\xi|^2} \\ \overset{\text{Inverse}}{\underset{\text{FT}}{\Longrightarrow}} \qquad w(x,t) &= (2\pi t)^{-n/2} \int f(x) e^{-\frac{1}{2t}|x-y|^2} \mathrm{d}y \\ &= f * p_t(x), \quad p_t(x) = (2\pi t)^{-n/2} e^{-\frac{1}{2t}|y|^2} \\ &= T_t f(x) \longrightarrow \text{semigroup.} \end{aligned}$$

But $p_t(y)$ = Gauß kernel = normal law = density of Brownian motion.



- Trap, x_0 absorbing = Killing = Dirichlet problem
- Reflect = Neumann problem
- Wait an go on = Sticky Brownian motion

Mind BM has cts paths, so it «sees» the boundary ∂D. And this is $\stackrel{1:1}{\leftrightarrow}$ generator is a local operator

Chapter 6

Examples: Jump-type (non-local) SDF_0

Need the Fourier transform

$$\hat{f}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} \mathrm{d}x$$
$$\check{g}(x) = \int_{\mathbb{R}^n} g(\xi) e^{i\xi\ x} \mathrm{d}\xi$$

So we have Plancherel[1]

$$\|f\|^2_{L^2(\mathrm{d}x)} = (2\pi)^{+n} \left\|\hat{f}\right\|^2_{L^2(\mathrm{d}\xi)}, \quad f \in L^2 \iff \hat{f} \in L^2$$

and

$$W^k(\mathbb{R}^n) = H^k(\mathbb{R}^n) \overset{\text{def}}{=} \left\{u \in L^2(\mathrm{d}x) : |\cdot|^k\, \hat{u} \in L^2(\mathrm{d}\xi)\right\}$$
$$= \Big\{u \in L^1_{\mathrm{loc}}(\mathrm{d}x) : \underbrace{\left(1 + |\cdot|^2\right)^{k/2}}_{\asymp 1+|\cdot|^k} \hat{u} \in L^2(\mathrm{d}\xi)\Big\}$$

Indeed:

$$u \in W^k(\mathbb{R}^n) \overset{\text{def}}{\iff} u \in L^2(\mathrm{d}x), \partial u^\alpha \in L^2(\mathrm{d}x), |\alpha| \leqslant k$$
$$\overset{\text{Plancherel}}{\iff} \quad \hat{u} \in L^2(\mathrm{d}\xi) = \widehat{\partial^\alpha u} = (i\xi)^\alpha \hat{u}(\xi) \in L^2(\mathrm{d}\xi), |\alpha| \leqslant k$$

and

$$c\left(\sum_{|\alpha|\leqslant k} |\xi^\alpha| + 1\right) \leqslant \left(1 + |\xi|^2\right)^{k/2} \leqslant c'\left(\sum_{|\alpha|\leqslant k} |\xi^\alpha| + 1\right)$$

jt-61 **6.1 Definition.** Let $s > 0$. Then

$$H^s(\mathbb{R}^n) = \left\{u \in L^2(\mathrm{d}x) : \left(1 + |\cdot|^2\right)^{s/2} \hat{u}(\cdot) \in L^2(\mathrm{d}\xi)\right\}$$

is called **fractional Sobolev space (order s)** or **Bessel-Potential space (order s)**.

Remark Read above as «$(1 - \Delta)^{s/2}\, u(x) \in L^2(\mathrm{d}x)$», and so, for $k = s \in \mathbb{N}$, $H^k = W^k$ and $(-\Delta)^{s/2} u = \mathfrak{F}^{-1}\left(|\xi|^s\ \hat{u}\right)$

[1]extend (by continuity) the Fourier transform onto L^2 (use L^2 complete.

jt-63 **6.2 Remark.** (a) $H^s(\mathbb{R}^n)$ is Hilbert with scalar product[2]

$$\langle u, v\rangle_{H^s} := \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \; \hat{u}\,\overline{\hat{v}}\,\mathrm{d}\xi$$

(b) $s>0$, $s=\lfloor s\rfloor+\{s\}$, $\lfloor s\rfloor\in\mathbb{N}_0$, $\{s\}\in(0,1)$. If $s\notin\mathbb{N}_0$:

$H^s(\mathbb{R}^n)=W^s(\mathbb{R}^n)=$ (Sobolev-)Slobodeckij space,

$$W^s(\mathbb{R}^n) := \left\{ u\in L^2(\mathrm{d}x) : u\in W^{\lfloor s\rfloor}(\mathbb{R}^n) \text{ and } \sum_{|\alpha|=\lfloor s\rfloor}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{n+2\{s\}}}\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\right.$$

Mind Read the above as some kind of Haar measure:

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left[\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^1}{|x-y|^{\{s\}}}\right]\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^n}\right)^{1/2}$$

(c) Interpolation («How to fill the gaps») W^k, $k=1,2,\dots$ are «natural», define $W^0:=L^2$. Then the **real interpolation** is[3]

$$\left(W^k,W^m\right)_{\vartheta,2}=W^s,\quad k<m,\ \vartheta\in(0,1),\ (1-\vartheta)k+\vartheta m=s,$$

a integral expression using a kernel. See [Trib] or [BS].

$$\left[W^k,W^m\right]_\vartheta=H^s,\quad \vartheta\in(0,1),\ s=(1-\vartheta)k+\vartheta m,$$

the **complex (Riesz-)Thorin interpolation or three-lines theorem from complex variables**.

(d) Using Plancherel again, we can give an alternative description fo the classical Dirichlet form:

$$\begin{aligned}
\mathcal{E}(u,v) &= \frac{(2\pi)^n}{2}\int_{\mathbb{R}^n}\widehat{\nabla u}\;\overline{\widehat{\nabla v}}\,\mathrm{d}\xi \qquad (u,v\in H^1)\\
&= \frac{(2\pi)^n}{2}\int_{\mathbb{R}^n}(i\xi)\overline{(i\xi)}\;\hat{u}(\xi)\overline{\hat{v}(\xi)}\,\mathrm{d}\xi\\
&= \frac{(2\pi)^n}{2}\int_{\mathbb{R}^n}|\xi|^2\,\hat{u}(\xi)\overline{\hat{v}(\xi)}\,\mathrm{d}\xi\\
&= \frac{1}{2}\langle-\Delta u, v\rangle_{L^2} \text{ if (say) } u,v\in C_c^\infty
\end{aligned}$$

jt-64 **6.3 Example.** Let $\alpha\in(0,2)$.

$$\mathcal{E}(u,v) := \frac{1}{2}(2\pi)^n\int_{\mathbb{R}^n}|\xi|^\alpha\;\hat{u}(\xi)\;\overline{\hat{v}(\xi)}\,\mathrm{d}\xi, \tag{6.1}$$ jt::eq01

[2] Exercise: Show completeness of H^s. Hints: $L^2(\mathrm{d}\xi)$ complete and Fatou + Resonance theorem.

[3] Mind the «2» as L^2 scale.

then $\left(\mathcal{E}, H^{\alpha/2}(\mathbb{R}^n)\right)$ is a symmetric SDF_0 and we have

$$\mathcal{E}(u,v) = \frac{1}{4}\int_{\mathbb{R}^n\setminus\{0\}}\int_{\mathbb{R}^n} (u(x+h)-u(x))\,(v(x+h)-v(x))\,\frac{c_\alpha\,\mathrm{d}x\,\mathrm{d}h}{|h|^{\alpha+n}}., \qquad (6.2)$$ jt::eq02

where $c_\alpha = \alpha\, 2^{\alpha-1}\pi^{-n/2}\,\Gamma\left(\frac{\alpha+n}{2}\right)/\Gamma\left(1-\frac{\alpha}{2}\right)$.[4]

Proof. We begin with (6.1) $\Longrightarrow$ (6.2). Need the Lévy-Khinchine formula[5]

$$|\xi|^\alpha = \int_{y\neq 0} (1-\cos y\xi)\,\frac{c_\alpha}{|y|^{\alpha+n}}\,\mathrm{d}y \qquad (6.3)$$ jt::eq03

Insert (6.3) into (6.1), assume $u, v \in C_c^\infty$ and use Fubini. Assume $u = v$ (get the full bilinear form by polarization).

$$\begin{aligned}
&\frac{1}{2}(2\pi)^n c_\alpha \int\int (1-\cos y\xi)\,|\hat{u}(\xi)|^2\,\frac{\mathrm{d}y}{|y|^{\alpha+n}}\,\mathrm{d}\xi \\
&= \frac{1}{2}(2\pi)^n c_\alpha \int_{y\neq 0}\frac{\mathrm{d}y}{|y|^{\alpha+n}}\int_{\mathbb{R}^n} |1-\cos y\xi|\,|\hat{u}(\xi)|^2\,\mathrm{d}\xi,
\end{aligned}$$

but the second integral gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\hat{u}|^2\,\mathrm{d}\xi - \left(\hat{u}\bar{\hat{u}}\right)^\vee(y) \\
&\overset{\text{Plancherel}}{\underset{\text{conv. thm}}{=}} (2\pi)^{-n}\int |u|^2\,\mathrm{d}x - (2\pi)^{-n}\hat{u}^\vee * \bar{\hat{u}}^\vee(y) \\
&= (2\pi)^{-n}\int |u|^2\,\mathrm{d}x - (2\pi)^{-n} u * \tilde{u}(y),
\end{aligned}$$

but $\tilde{u}(y) = u(-y)$ is just a reflection. Moreover,

$$\begin{aligned}
&= \frac{1}{2}c_\alpha \int \frac{\mathrm{d}y}{|y|^{\alpha+n}}\int u^2(x)\,\mathrm{d}x - \int u(-x)u(y-x)\,\mathrm{d}x \\
&= \frac{1}{2}c_\alpha \int_{y\neq 0}\int_{\mathbb{R}^n} u(x)\,(u(x)-u(x+y))\,\frac{\mathrm{d}x\,\mathrm{d}y}{|y|^{\alpha+n}} \\
&= \underbrace{\frac{1}{4}c_\alpha \iint}_{\text{keep}} + \underbrace{\frac{1}{4}c_\alpha \iint}_{x\to x-y,\ y\to -y} \\
&= \frac{1}{4}c_\alpha \int_{y\neq 0}\int_{\mathbb{R}^n} (u(x)-u(x+y))^2\,\frac{\mathrm{d}x\,\mathrm{d}y}{|y|^{\alpha+n}}
\end{aligned}$$

SDF_0 $(\mathcal{E}_1)$, $\gamma = 0$, clear by (6.1). $(\mathcal{E}_2)$ clear b/o symmetry, (6.2). $(\mathcal{E}_3)$ clear as $\mathcal{E}_1$ is scalar product in $H^{\alpha/2}$. $(\mathcal{E}_4)$ is clear by (6.2), since Lipschitz functions operator on differences.

Regularity C_c^∞ dense in $H^{\alpha/2}$. This follows from:

[4] Exercise: Change $x - y = h$, compare with W^s.

[5] The bracket is $\approx |y|^2$ if $|y| < 1$ and $\leqslant 2$, if $|y| \geqslant 1 \Longrightarrow$ integrable

1° $C_c^\infty \subset \mathcal{S}$ = Schwartz rapidly decreasing functions dense in L^2

2° $u \in C_c^\infty \Longrightarrow \hat{u} \in \mathcal{S}$, $\left(1 + |\xi|^2\right)^{\alpha/2} \hat{u}(\xi) \in \mathcal{S}(\mathbb{R}^n)$.

$\Longrightarrow \mathcal{S}$ dense in $H^{\alpha/2}$.

Generator of $\mathcal{E}(u, v)$ Define

$$\begin{aligned}(-\Delta)^{\alpha/2} u(x) &:= \int |\xi|^\alpha \, \hat{u}(\xi) e^{ix\xi} \, \mathrm{d}\xi \\ &= \mathcal{F}^{-1}\left(|\cdot|^\alpha \, \hat{u}\right)(x).\end{aligned}$$

$|\xi|^\alpha$ is called the symbol of the Pseudo-differential operator $(-\Delta)^{\alpha/2}$,

$$\begin{aligned}\mathcal{E}(u, v) &= \frac{1}{2}(2\pi)^n \int \{|\xi|^\alpha \ \hat{u}(\xi)\} \, \overline{\hat{v}(\xi)} \, \mathrm{d}\xi \\ \frac{1}{2}(-\Delta)^{\alpha/2} u(x) &:= \frac{1}{2} \int |\xi|^\alpha \ \hat{u}(\xi) e^{ix\,\xi} \, \mathrm{d}\xi \\ &= (2\pi)^n \int \frac{1}{2} \widehat{(-\Delta)^{\alpha/2} u} \, \overline{\hat{v}} \, \mathrm{d}\xi \\ &\overset{\text{Plan}}{=} \left\langle \frac{1}{2}(-\Delta)^{\alpha/2} u, v \right\rangle_{L^2} \quad \text{is its generator.}\end{aligned}$$

Afterwards justify all is ok for $u \in H^\alpha, v \in H^{\alpha/2}$. ∎

The semigroup associated with $\mathcal{E}$ is a convolution semigroup, α-stable, leads to symmetric α-stable Lévy processes.

6.4 Remark (Outlook). Consider $\psi : \mathbb{R}^n \to \mathbb{C}$. Then we have the **Lévy Khinchine formula**

$$\psi(\xi) = il\xi + \frac{1}{2}\mathbf{Q}\xi \cdot \xi + \int_{y \neq 0} \left[1 - e^{iy\xi} + iy\xi \mathbb{1}_{(0,1)}(|y|)\right] \nu(\mathrm{d}y), \tag{6.4}$$ jt::eq04

where $l \in \mathbb{R}^n$, $\mathbf{Q} \in \mathbb{R}^{n \times n}$ symmetric, positive $\frac{1}{2}$-definite, and ν is a measure on $\mathbb{R}^n \setminus \{0\}$ with $\int \left(|y|^2 \wedge 1\right) \nu(\mathrm{d}y) < \infty$, which characterizes *all possible* Lévy processes.

Note $\psi(\xi) = |\xi|^\alpha \longrightarrow l = 0, \ \mathbf{Q} = 0, \ \nu(\mathrm{d}y) = \dfrac{c_\alpha \, \mathrm{d}y}{|y|^{n+\alpha}}$

$$\mathcal{E}(u, v) = (2\pi)^n \int \psi(\xi) \hat{u}(\xi) \, \overline{\hat{v}(\xi)} \, \mathrm{d}\xi$$

is a SDF_0 if the following «sector condition» holds:

$$|\mathfrak{I}(\psi)| \leqslant \text{const } \mathfrak{R}\psi(\xi),$$

which gives $(\mathcal{E}_2)$. $\mathcal{E}$ is symmetric $\iff$ ψ is real. So there are Lévy processes which does not have a Dirichlet form (Berg + Forst ≈ 1973).

Domain $\mathcal{F} = H^{\psi,1} = \left\{ u \in L^2 : (1 + |\psi(\xi)|)^{1/2}\, \hat{u} \in L^2 \right\}.$

Chapter 7

Excessive Functions

Setting $(\mathcal{E}, \mathcal{F})$ SDF_γ, semigroup $(T_t)_{t\geqslant 0}$, resolvent $(\mathbf{G}_\alpha)\alpha > \gamma$, recall the «hattet» objects given by

$$\hat{\mathcal{E}}(u, v) := \mathcal{E}(v, u) \qquad \text{(«dual form»)}$$

ef-01 **7.1 Definition.** $\alpha > 0$, $u \in L^2(m)$ is called α-**excessive [α-co-excessive]**, notation $\mathrm{Exc}(\alpha)$ [$\mathrm{Ex\hat{c}}(\alpha)$], if

$$\begin{aligned} e^{-\alpha t} T_t u &\leqslant u \quad m-\text{a.s. } \forall t \geqslant 0, \\ \left[e^{-\alpha t} \hat{T}_t u \right. &\leqslant u \quad m-\text{a.s. } \left.\forall t \geqslant 0\right], \end{aligned} \tag{7.1}$$ ef::eq01

ef-02 **7.2 Theorem.** *Let $u \in \mathcal{F}$, $\alpha > \gamma$. TFAE:*

(a) u is α-excessive

(b) $\beta \mathbf{G}_{\alpha+\beta} u \leqslant u$ m-a.e. $\forall \beta > 0$

(c) $\mathcal{E}_\alpha(u, v) \geqslant 0$ $\forall v \in \mathcal{F}^+ = \{w \in \mathcal{F} : w \geqslant 0\}$

Dual for the «hat versions».

Proof. (a) $\Longrightarrow$ (b) (holds even for $u \in L^2(m)$)

$$\beta \mathbf{G}_{\alpha+\beta} u = \int_0^\infty \beta\, e^{-(\alpha+\beta)t} T_t u \,\mathrm{d}t \leqslant \int_0^\infty \beta\, e^{-\beta t} \,\mathrm{d}t \cdot u$$

(b) $\Longrightarrow$ (c) Use approximating form $\mathcal{E}_\beta$ of § 3 and

$$\begin{aligned} \mathcal{E}_\alpha(u, v) &= \lim_{\beta\to\infty} \mathcal{E}^\beta(u, v) + \alpha \langle u, v \rangle_{L^2(m)} \\ &\overset{\text{def}}{=} \lim_{\beta\to\infty} \beta \left\langle u - \beta \mathbf{G}_\beta u \mathbf{G}_{\alpha+\beta} u) - \beta \mathbf{G}_{\beta+\alpha} u, v \right\rangle_{L^2(m)} + \alpha \langle u, v \rangle_{L^2(m)}, \\ &= \lim_{\beta\to\infty} \left(\beta \left\langle u - \beta \mathbf{G}_{\beta+\alpha} u, v \right\rangle_{L^2(m)} + \alpha \left\langle u - \beta^2 \mathbf{G}_\beta \mathbf{G}_{\alpha+\beta} u, v \right\rangle_{L^2(m)} \right) \\ &= \lim_{\beta\to\infty} \left(\beta \left\langle u - \beta \mathbf{G}_{\beta+\alpha} u, v \right\rangle_{L^2(m)} + \alpha \left\langle u - \beta \mathbf{G}_\beta u, v \right\rangle_{L(^2)} + \right. \\ &\qquad \left. \alpha \langle \underbrace{\beta \mathbf{G}_\beta u - \beta \mathbf{G}_{\beta+\alpha} \beta \mathbf{G}_\beta u}_{= \beta \mathbf{G}_\beta (u - \beta \mathbf{G}_{\alpha+\beta} u) \geqslant 0}, v \rangle_{L^2(m)} \right), \end{aligned}$$

so the first and third term are positive by (b) and the second term goes to 0 b/o $\beta \mathbf{G}_\beta \to \mathrm{id}$ $(\beta \to 0)$.

(c) $\Longrightarrow$ (a) Pick $v \in L^2_+(m)$.

$$\hat{\mathbf{G}}_\alpha v - e^{-\alpha t}\hat{T}_t \hat{\mathbf{G}}_\alpha v \overset{\text{def}}{=} \int_0^t e^{-\alpha s}\hat{T}_s v \, \mathrm{d}s \geqslant 0$$

and so by (c),

$$\begin{aligned}\left\langle u - e^{-\alpha t}T_t u, v\right\rangle_{L^2} &= \left\langle u, v - e^{-\alpha t}\hat{T}_t v\right\rangle_{L^2} \\ &\overset{(??)}{=} \mathcal{E}_\alpha(u, \hat{\mathbf{G}}_\alpha v - e^{-\alpha t}\hat{\mathbf{G}}_\alpha \hat{T}_t v)_{L^2} \overset{(c)}{\geqslant} 0\end{aligned}$$

and so $u - e^{-\alpha t}T_t u \geqslant 0$ as v arbitrary $\geqslant 0$.

■

ef-73 **7.3 Remark.** 1. 7.2 (a) $\Longrightarrow$ (b) only needs $u \in L^2(m)$

2. $u \in \mathrm{Exc}(\alpha) \Longrightarrow u \geqslant 0$

Indeed Using sub-Markov,

$$\left\| e^{-\alpha t} T_t u \right\|_{L^2(m)} \leqslant e^{-\alpha t} \, \|u\|_{L^2(m)}$$

7.4 Remark (Properties of $\mathrm{Exc}(\alpha)$). Let $\alpha > \gamma$.

(a) $f \in L^2_+(m) \Longrightarrow \mathbf{G}_\alpha f \in \mathrm{Exc}(\alpha)$

(b) $\mathfrak{F} \cap (\mathrm{Exc}(\alpha) - \mathrm{Exc}(\alpha)) = \mathfrak{F} \cap \{f - g \,:\, f, g \in \mathrm{Exc}(\alpha)\}$ dense in $(\mathfrak{F}, \mathcal{E}^s_\alpha)$

(c) $u \in \mathrm{Exc}(\alpha), u \leqslant v \in \mathfrak{F} \Longrightarrow u \in \mathfrak{F}$

(d) $u \in \mathrm{Exc}(\alpha) \cap \mathfrak{F} \Longrightarrow \mathbf{G}_\lambda \in \mathrm{Exc}(\alpha)\ \forall \lambda > \gamma$

(e) $f, g \in \mathrm{Exc}(\alpha) \Longrightarrow f \wedge g \in \mathrm{Exc}(\alpha)$

(f) $\mathrm{Exc}(\alpha) \subset \mathrm{Exc}(\alpha + \beta)\ \forall \beta > 0$

(g) $f \in \mathrm{Exc}(\alpha) \Longrightarrow \beta \mathbf{G}_{\alpha+\beta} f \nearrow (\beta \uparrow)$

(h) $\mathrm{Exc}(\alpha) + \mathrm{Exc}(\alpha) \subset \mathrm{Exc}(\alpha)$

Proof. (a) Since $\mathbf{G}_\alpha f \in \mathfrak{F}$, we may use Proposition 7.2 or

$$\beta \mathbf{G}_{\alpha+\beta}\mathbf{G}_\alpha f \underset{\text{eqn}}{\overset{\text{res}}{=}} (\mathbf{G}_\alpha - \mathbf{G}_{\alpha+\beta}) f \leqslant \mathbf{G}_\alpha f .$$

(b) $\mathcal{F} \cap (\mathrm{Exc}(\alpha) - \mathrm{Exc}(\alpha)) \overset{(a)}{\supset} \mathbf{G}_\alpha(L^2(m)) \cap \mathcal{F} \overset{L3.6}{=} \mathfrak{D}(\mathbf{A}) \cap \mathcal{F} = \mathfrak{D}(\mathbf{A})$ dense in $\mathcal{F}$

(c) By Theorem 3.5, we show $\mathcal{E}^\beta(u,u)$ bounded for $\beta > 0$. So by CSI and $\|\mathbf{G}_\delta\| \leqslant \frac{1}{\delta-\gamma}$

$$\begin{aligned}
\mathcal{E}^\beta(u,u) &\overset{\text{def}}{=} \beta \left\langle u - \beta \mathbf{G}_\beta u, u \right\rangle_{L^2(m)} \\
&= \beta \left\langle u - \beta \mathbf{G}_{\beta+\alpha} u, u \right\rangle_{L^2(m)} + \beta^2 \left\langle \mathbf{G}_{\beta+\alpha} u - \mathbf{G}_\beta u, u \right\rangle_{L^2(m)} \\
&\leqslant \beta \left\langle u - \beta \mathbf{G}_{\beta+\alpha} u, v \right\rangle_{L^2(m)} + \beta^2 \left\langle \mathbf{G}_{\beta+\alpha} u - \mathbf{G}_\beta u, u \right\rangle_{L^2(m)} \\
&\overset{\text{def}}{=} \mathcal{E}^\beta(u,v) + \beta^2 \langle \underbrace{\mathbf{G}_{\alpha+\beta} u - \mathbf{G}_\beta u}_{= -\alpha \mathbf{G}_\alpha \mathbf{G}_{\alpha+\beta} u}, u - v \rangle_{L^2(m)} \\
&\overset{(??)}{\leqslant} \kappa \sqrt{\mathcal{E}^\beta_\lambda(u)} \sqrt{\mathcal{E}_\gamma(v)} + \beta^2 \alpha \frac{1}{\beta - \gamma} \frac{1}{\alpha + \beta - \gamma} \|u\|_{L^2(m)} \|u - v\|_{L^2(m)},
\end{aligned}$$

ok for all $\lambda > \gamma \left(\frac{\beta}{\beta-\gamma}\right)^2$, cf. (??). So for $\beta \gg \gamma$ we get boundedness of the lhs.[1] Hence, $u \in \mathcal{F}$.

(d)

$$\beta \mathbf{G}_{\alpha+\beta} \mathbf{G}_\lambda u = \underbrace{\mathbf{G}_\lambda}_{\text{positive}} \underbrace{(\beta \mathbf{G}_{\alpha+\beta} u)}_{\leqslant u} \leqslant \mathbf{G}_\lambda u \quad \forall \beta > 0,$$

apply Proposition 7.2 $\Longrightarrow \mathbf{G}_\lambda u \in \mathrm{Exc}(\alpha)$.

(e), (f), (g), (h) Exercise. Hint for (e): $e^{-\alpha t} T_t f \leqslant f$, $e^{-\alpha t} T_t g \leqslant g$ and $T_t(f \wedge g) \leqslant T_t f, T_t g \curvearrowright e^{-\alpha t} T_t (f \wedge g) \leqslant f, g$. Hint for (f) $e^{-\alpha t} \leqslant 1$. Hint for (g) Resolvent equation.

■

[1] See arguement at the end of the proof of (??) in Lemma 3.4.

Chapter 8

Capacity

Problem $\exists$ more then $\sharp\mathbb{N}$ exceptional sets, so «m-a.e.», m-null sets not good enough.

Way out measure $\longrightarrow$ capacity,
a.e. $\longrightarrow$ quasi-everywhere q.e. = outside a set of capacity 0

Setting $(\mathcal{E},\mathfrak{F})$ SDF$_\gamma$, $\alpha>\gamma$, assume $(\mathcal{E}_1)-(\mathcal{E}_3)$, $(\mathcal{E}_4)$, but $\left(\hat{\mathcal{E}}_4\right)$ not always required.

Idea Need a projection

cap-81 **8.1 Definition.** $\varnothing\neq\Gamma\subset\mathfrak{F}$, convex, closed, $\alpha>\gamma$. An α-**(co)projection** of $u\in\mathfrak{F}$ is any $v\in\Gamma$ $[\hat{v}\in\Gamma]$ with

$$\forall w\in\Gamma:\ \mathcal{E}_\alpha(u-v,w-v)\leqslant 0 \tag{8.1}$$ cap::eq01

$$\forall w\in\Gamma:\ \hat{\mathcal{E}}_\alpha(u-\hat{v},w-\hat{v})\leqslant 0 \tag{8.\^1}$$ cap::eq01h

Notation: $v=\pi_\Gamma^\alpha(u)$, $\hat{v}=\hat{\pi}_\Gamma^\alpha(u)$.

Problem well-defined?

cap-82 **8.2 Lemma.** *$\pi_\Gamma^\alpha(u)$, $\hat{\pi}_\Gamma^\alpha(u)$ exists and is unique ($u\in\mathfrak{F}$).*

Proof. Rewrite (8.1) as

$$\forall w\in\Gamma:\ \underbrace{\mathcal{E}_\alpha(u,w-v)}_{=:\mathfrak{J}(w-v)}\leqslant\mathcal{E}_\alpha(v,w-v), \tag{8.2}$$ cap::eq02

but $\mathfrak{J}$ is a linear functional, $\mathcal{E}_\alpha$-cts, since u is fixed. Hence, by Stampacchia's theorem, $\exists!$ unique solution v of (8.2). «Hat versions» analoguously. ■

We need special Γs and π_Γ^αs.

cap-83 **8.3 Definition.** $A\subset X$ Borel set.[1]

$$\mathcal{L}_A^{=}:=\left\{w\in\mathfrak{F}:\ w\big|_A\overset{=}{\geqslant}1\ m\text{-a.e.}\right\} \tag{8.3}$$ cap::eq03

[1]Exercise: $\mathcal{L}_A^{=}$ convex, closed in $\mathfrak{F}$. Hint: Use a subsequence and a.e. convergence. In the classical and symmetric case $\mathcal{L}_A=\mathcal{L}_A^{=}$

cap-84 **8.4 Definition.** $e_A^\alpha := \pi_{\mathcal{L}}^\alpha(0)\,[\hat{e}_A^\alpha := \hat{\pi}_{\mathcal{L}}^\alpha(0)]\,(0 \in \mathcal{F})$ is the α-**(co)equilibrium potential** of $A \subset X$, A Borel.

cap-85 **8.5 Lemma** (Properties of e_A). *$A \subset X$ Borel.*

(a) $\mathcal{E}_\alpha(e_A^\alpha) \leqslant \mathcal{E}_\alpha(e_A^\alpha, w)\ \forall w \in \mathcal{L}_A$, *Dual for $\wedge$-version*

(b) $\mathcal{E}_\alpha(e_A^\alpha) \leqslant \kappa_\alpha^2 \mathcal{E}_\alpha(w)\ \forall w \in \mathcal{L}_A$, *Dual for $\wedge$-version*

(c) $0 \leqslant e_A^\alpha \leqslant 1$ *and* $e_A^\alpha\big|_A = 1$ *m-a.e., but* $\hat{e}_A^\alpha \geqslant 0$ *and* $\hat{e}_A^\alpha\big|_A \leqslant 1$ *(b/o do not have* $(\hat{\mathcal{E}}_4)$*, only* $(\mathcal{E}_4)$*).*

(d) $u \in \mathcal{F}$, $u\big|_A = 1$ *m-a.e., then*

$$\mathcal{E}_\alpha(e_A^\alpha, u) = \mathcal{E}_\alpha(e_A^\alpha)$$
$$\hat{\mathcal{E}}_\alpha(\hat{e}_A^\alpha, u) = \hat{\mathcal{E}}_\alpha(\hat{e}_A^\alpha, e_A^\alpha)$$

(e) $\mathcal{E}_\alpha(e_A^\alpha) = 0 \Longrightarrow m(A)$*. Dual for $\wedge$-version*

(f) $e_A^\alpha \in \mathrm{Exc}(\alpha)$, $\hat{e}_A^\alpha \in \widehat{\mathrm{Exc}}(\alpha)$

(g) $A \subset B$, $e_A^\alpha \leqslant e_B^\alpha$, *Dual for $\wedge$-version*

Proof. (a) $v = \pi_\Gamma^\alpha(0)$, $u = 0$, $\Gamma = \mathcal{L}_A$, then (8.1) gives

$$\mathcal{E}_\alpha(0 - e_A^\alpha, w - e_A^\alpha) \leqslant 0 \quad \forall w \in \mathcal{L}_A.$$

(b)

$$\begin{aligned}\mathcal{E}_\alpha(e_A^\alpha) &\underset{w \in \mathcal{L}_A}{\overset{\text{(a)}}{\leqslant}} \mathcal{E}_\alpha(e_A^\alpha, w) \\ &\overset{\text{sector}}{\leqslant} \kappa_\alpha \sqrt{\mathcal{E}_\alpha(e_A^\alpha)}\sqrt{\mathcal{E}_\alpha(w)}\end{aligned}$$

(c) **Show** $e_A^\alpha \overset{?}{=} \underbrace{(e_A^\alpha)^+ \wedge 1}_{\in \mathcal{F} \text{ b/o } (\mathcal{E}_4')} \in \mathcal{L}_A$, as $(e_A^\alpha)^+ \wedge 1 = 1$ on A

$$\begin{aligned}0 &\leqslant \mathcal{E}_\alpha\left((e_A^\alpha)^+ \wedge 1 - e_A^\alpha)\right) \\ &= \underbrace{\mathcal{E}_\alpha((e_A^\alpha)^+ \wedge 1, (e_A^\alpha)^+ \wedge 1 - e_A^\alpha)}_{\leqslant 0 \text{ b/o } (\mathcal{E}_4')} - \underbrace{\mathcal{E}_\alpha(e_A^\alpha, \underbrace{(e_A^\alpha)^+ \wedge 1}_{\in \mathcal{L}_A} - e_A^\alpha)}_{\geqslant 0 \text{ by (a)}} \\ &\leqslant 0\end{aligned}$$

So $(e_A^\alpha)^+ \wedge 1 = e_A^\alpha$ as functions of $\mathcal{F}$, hence m-a.e.

(d) $v \in \mathcal{F}, v\big|_A \geqslant 0 \Longrightarrow v + e_A^\alpha\big|_A \geqslant 1.$

$$\begin{aligned}
&\overset{\text{(a)}}{\Longrightarrow} && \mathcal{E}_\alpha(e_A^\alpha) \leqslant \mathcal{E}_\alpha(e_A^\alpha, \underbrace{e_A^\alpha + v}_{\cong w \in \mathcal{L}_A}) \\
&\Longrightarrow && 0 \leqslant \mathcal{E}_\alpha(e_A^\alpha, v) \quad \forall v \in \mathcal{F}, v\big|_A \geqslant 0 \\
&\overset{\pm v}{\Longrightarrow} && 0 = \mathcal{E}_\alpha(e_A^\alpha, v) \quad \forall v \in \mathcal{F}, v\big|_A = 0, \text{ same for } \hat{\mathcal{E}}_\alpha.
\end{aligned}$$

If $u \in \mathcal{F}$ and $u\big|_A = 1$, then $v := e_A^\alpha - u \in \mathcal{F}$ and $v\big|_A = 0 \Longrightarrow$ gives formulae.[2]

(e) Trivial as $\mathcal{E}_\alpha^s$ $(\alpha > \gamma)$ as a scalar product (+ (c)).

(f) $v \in \mathcal{F}, v \geqslant 0 \Longrightarrow v + e_A^\alpha\big|_A \geqslant 1.$

$$\begin{aligned}
&\overset{\text{(a)}}{\Longrightarrow} && \mathcal{E}_\alpha(e_A^\alpha, v + e_A^\alpha) \geqslant \mathcal{E}_\alpha(e_A^\alpha, e_A^\alpha) \\
&\Longrightarrow && \mathcal{E}_\alpha(e_A^\alpha, v) \geqslant 0 \\
&\overset{\text{7.2 (c)}}{\Longrightarrow} && e_A^\alpha \text{ excessive}
\end{aligned}$$

Same for $\hat{e}_A^\alpha$ co-excessive $\overset{??}{\Longrightarrow} \hat{e}_A^\alpha \geqslant 0 \Longrightarrow$ proves 2nd line of (c).

(g) $A \subset B$. We show $e_A^\alpha \wedge e_B^\alpha = e_A^\alpha$. Know, $e_A^\alpha, e_B^\alpha \in \operatorname{Exc}(\alpha) \overset{??}{\Longrightarrow} e_A^\alpha \wedge e_A^\alpha \in \operatorname{Exc}(\alpha)$. Hence

$$\mathcal{E}_\alpha(e_A^\alpha \wedge e_B^\alpha, e_A^\alpha - e_A^\alpha \wedge e_B^\alpha) \overset{??\ c}{\geqslant} 0 \tag{$*$}$$

But (a)

$$\begin{aligned}
&\Longrightarrow && \mathcal{E}_\alpha(e_A^\alpha, e_A^\alpha \wedge e_B^\alpha) - \mathcal{E}_\alpha(e_A^\alpha, e_A^\alpha) \geqslant 0 \\
&\Longleftrightarrow && \mathcal{E}_\alpha(-e_A^\alpha, e_A^\alpha - e_A^\alpha \wedge e_B^\alpha) \geqslant 0
\end{aligned} \tag{$**$}$$

Now add $(*)$ and $(**)$

$$\begin{aligned}
& && \mathcal{E}_\alpha(e_A^\alpha \wedge e_B^\alpha - e_A^\alpha, e_A^\alpha - e_A^\alpha \wedge e_B^\alpha) \geqslant 0 \\
&\Longrightarrow && 0 \leqslant \mathcal{E}_\alpha(e_A^\alpha - e_A^\alpha \wedge e_B^\alpha, e_A^\alpha - e_A^\alpha \wedge e_B^\alpha) \leqslant 0 \\
&\Longrightarrow && e_A^\alpha = e_A^\alpha \wedge e_B^\alpha
\end{aligned}$$

■

Remark 8.5 (b) yields $(e_A^\alpha, \hat{e}_A^\alpha \in \mathcal{L}_A)$

$$\mathcal{E}_\alpha(e_A^\alpha) \leqslant \kappa_\alpha^2 \mathcal{E}_\alpha(\hat{e}_A^\alpha) \leqslant \kappa^4 \mathcal{E}_\alpha \mathcal{E}_\alpha(e_A^\alpha) \tag{$\sharp$}$$

[2] Attention: of we know that $\hat{e}_A^\alpha\big|_A = 1$, then same trick work with $v = \hat{e}_A^\alpha - u$ and more results, $\mathcal{L}_A^=$.

cap-86 **8.6 Definition.** $U \subset X$ open. Then the α-**capacity**

$$\mathrm{cap}_\alpha(U) = \begin{cases} \mathcal{E}_\alpha(e^\alpha_U, \hat{e}^\alpha_U) & \mathcal{L}_U \neq \varnothing \\ +\infty & \mathcal{L}_U = \varnothing \end{cases} \tag{8.4}$$ cap::eq0

($\sharp$) follows[3]

$$\mathrm{cap}_\alpha(U) \asymp \mathcal{E}_\alpha(\hat{e}^\alpha_U) \asymp \mathcal{E}_\alpha(e^\alpha_U) \tag{8.5}$$ cap::eq0

cap-87 **8.7 Lemma** (Properties of cap_α). *$U_n, U, V \subset X$ open, $\alpha > \gamma$.*

(a) $U \subset V \implies \mathrm{cap}_\alpha(U) \leqslant \mathrm{cap}_\alpha(V)$ *(monotone)*

(b) $\mathrm{cap}_\alpha(U \cup V) + \mathrm{cap}_\alpha(U \cap V) \leqslant \mathrm{cap}_\alpha(U) + \mathrm{cap}_\alpha(V)$ *(strong sub-additive)*

(c) $U_n \uparrow U = \bigcup_{n\in\mathbb{N}} U_n \implies \mathrm{cap}_\alpha(U_n) \uparrow \mathrm{cap}_\alpha(U)$ *(continuity from below)*

Proof. (a)

$$\begin{aligned} \mathrm{cap}_\alpha(U) &\overset{\text{def}}{=} \mathcal{E}_\alpha(e^\alpha_U, \hat{e}^\alpha_U) \\ &\underset{e^\alpha_U \in \mathrm{Exc}(\alpha)}{\overset{\hat{e}^\alpha_U \leqslant \hat{e}^\alpha_V}{\leqslant}} \mathcal{E}_\alpha(e^\alpha_U, \hat{e}^\alpha_V) \\ &\underset{\hat{e}^\alpha_V \in \widehat{\mathrm{Exc}}(\alpha)}{\overset{e^\alpha_U \leqslant e^\alpha_V}{\leqslant}} \mathcal{E}_\alpha(e^\alpha_V, \hat{e}^\alpha_V) = \mathrm{cap}_\alpha(V) \end{aligned}$$

(b) **Claim** $\hat{e}^\alpha_{U\cup V} \leqslant \hat{e}^\alpha_U + \hat{e}^\alpha_V$. Indeed: $W := U \cup V \iff \hat{e}^\alpha_W = \hat{e}^\alpha_W \wedge (\hat{e}^\alpha_U + \hat{e}^\alpha_V)$.

$$1^{\circ}\ \hat{\mathcal{E}}_\alpha\Big(\underbrace{\underbrace{\hat{e}^\alpha_W}_{\widehat{\mathrm{Exc}}(\alpha),8.5\mathrm{g}} \wedge \underbrace{(\hat{e}^\alpha_U + \hat{e}^\alpha_V)}_{\widehat{\mathrm{Exc}}(\alpha),8.5\mathrm{g},\,\mathrm{h}}}_{\widehat{\mathrm{Exc}}(\alpha),7.4\ \mathrm{e}}, \underbrace{\hat{e}^\alpha_W - \hat{e}^\alpha_W \wedge (\hat{e}^\alpha_U + \hat{e}^\alpha_V)}_{\geqslant 0,\ \mathcal{F}}\Big) \geqslant 0$$

2° $\hat{\mathcal{E}}_\alpha(\hat{e}^\alpha_W) - \hat{\mathcal{E}}_\alpha(\hat{e}^\alpha_W, \hat{e}^\alpha_W \wedge (\hat{e}^\alpha_U + \hat{e}^\alpha_V)) \leqslant 0$ by 8.5 (a).

Now substract 1° - 2°

$$\begin{aligned} &\implies & 0 \leqslant \hat{\mathcal{E}}_\alpha(\hat{e}^\alpha_W \wedge (\hat{e}^\alpha_U + \hat{e}^\alpha_V) - \hat{e}^\alpha_W) \leqslant 0 \\ &\implies & \text{claim} \quad \hat{e}^\alpha_W = \hat{e}^\alpha_{U\cup V} \leqslant \hat{e}^\alpha_U + \hat{e}^\alpha_V \end{aligned}$$

[3] $\asymp$ means comparable with absolute constants w.r.t U.

Now

$$\operatorname{cap}_\alpha(U\cup V)+\operatorname{cap}_\alpha(U\cap V)\overset{\text{def}}{=}\mathcal{E}_\alpha(e^\alpha_{U\cup V})+\mathcal{E}_\alpha(\underbrace{e^\alpha_{U\cap V}}_{\operatorname{Exc}(\alpha)},\underbrace{\hat e^\alpha_{U\cap V}}_{\leqslant \hat e^\alpha_{U\operatorname{cap}V}})$$

$$\leqslant \mathcal{E}_\alpha(e^\alpha_{U\cup V},\hat e^\alpha_{U\cup V})+\mathcal{E}_\alpha(e^\alpha_{U\cap V},\hat e^\alpha_{U\cap V})$$

$$\leqslant \mathcal{E}_\alpha(\underbrace{u^\alpha_{U\cap V}+e^\alpha_{U\cap V}}_{\geqslant 1\text{ on }U,V},\hat e^\alpha_U+\hat e^\alpha_V)$$

$$\overset{8.5\ (d)}{=}\mathcal{E}_\alpha(e^\alpha_U,\hat e^\alpha_U)+\mathcal{E}_\alpha(e^\alpha_V,\hat e^\alpha_V)\overset{\text{def}}{=}\operatorname{cap}_\alpha(U)+\operatorname{cap}_\alpha(V)$$

Fix

$$\operatorname{cap}(U\cup V)\leqslant \operatorname{cap}_\alpha(U)+\operatorname{cap}_\alpha(V).$$

Proof. 1° $\hat e^\alpha_{U\cup V}\leqslant \hat e^\alpha_U e+\hat e^\alpha_V$.

Further

$$\operatorname{cap}_\alpha(U\cup V)\overset{\text{def}}{=}\mathcal{E}_\alpha(\underbrace{e^\alpha_{U\cup V}}_{\operatorname{Exc}(\alpha)},\underbrace{\hat e^\alpha_{U\cup V}}_{\leqslant \hat e^\alpha_U+\hat e^\alpha_V})$$

$$\overset{\text{Exc}}{\leqslant}\mathcal{E}_\alpha(e^\alpha_{U\cup V})$$

$$\overset{8.5\ d)}{=}\mathcal{E}_\alpha(e^\alpha_U,\hat e^\alpha_U)+\mathcal{E}_\alpha(e^\alpha_V,\hat e^\alpha_V)=\operatorname{cap}_\alpha(U)+\operatorname{cap}_\alpha(V).$$

■

(c) **Always ok** $\sup_n \operatorname{cap}_\alpha(U_n)\leqslant \operatorname{cap}_\alpha(U)$. WLOG assume $\sup_n \operatorname{cap}_\alpha(U_n)<\infty$.

$$\infty>\sup_n \operatorname{cap}_\alpha(U_n)=\sup_n \mathcal{E}_\alpha(e^\alpha_{U_n},\hat e^\alpha_{U_n})$$

$$\asymp \sup_n \mathcal{E}_\alpha(\hat e^\alpha_{U_n})$$

Then (see FA refresher) $\exists \hat e\in\mathcal{F}\ \exists n(k):\ \hat e_{U_{n(k)}}\rightharpoonup \hat e$, even $\hat e_{U_{n(k)}}\uparrow \hat e$ m-a.e. and $\hat e\in\mathcal{L}_U$. Then[4], for any $w\in\mathcal{L}$,

$$\mathcal{E}_\alpha(w,\hat e)=\lim_{k\to\infty}\mathcal{E}_\alpha(w,\hat e^\alpha_{U_{n(k)}})$$

$$\underset{w\in\mathcal{L}_U}{\overset{8.5}{\geqslant}}\liminf_k \mathcal{E}_\alpha(\hat e^\alpha_{U_{n(k)}}$$

$$\overset{\text{resonance}}{\geqslant}\mathcal{E}_\alpha(\hat e)$$

[4]Exercise

Hence, $\hat{e} = \hat{e}_U$ by uniqueness of $\hat{e}_U$. Moreover

$$\begin{aligned} \operatorname{cap}_\alpha(U) &= \mathcal{E}_\alpha(e_U, \hat{e}_U) = \lim_k \mathcal{E}_\alpha(e_U^\alpha, \hat{e}^\alpha_{U_{n(k)}}) \\ &\overset{8.5\,(d)}{=} \lim_k \mathcal{E}_\alpha(e^\alpha_{U_{n(k)}}, \hat{e}^\alpha_{U_{n(k)}}) \\ &= \lim_k \operatorname{cap}_\alpha(U_{n(k)}). \end{aligned}$$

Since $\operatorname{cap}_\alpha$ is monotone $\Longrightarrow$ subsequence does not matter.

■

Exercises

(1) $u_n \rightharpoonup u \Longrightarrow \mathcal{E}_\alpha(u_n, w) \to \mathcal{E}_\alpha(u, w)\ \forall w \in \mathfrak{F}$. Hint: linear, sector condition, use Riesz.

(2) 8.7 ok for all Borel sets.

(3) 8.7 (c) ok for any $\mathbf{A}_\lambda \uparrow \mathbf{A}$, $\mathbf{A}_\lambda \uparrow \mathbf{A}$.

(4) As for measures we have 8.7 b) + 8.7 c) $\Longrightarrow \operatorname{cap}_\alpha(\bigcup_{n\in\mathbb{N}} U_n) \leqslant \sum_{n\in\mathbb{N}} \operatorname{cap}_\alpha(U_n)$.

(5) 8.7 holds for Borel sets (not only for open sets).

Now Standard procedure (Choquet, ~50-60) to extend $\operatorname{cap}_\alpha\big|_{\mathfrak{O}}$, $\mathfrak{O}$ the open sets

```
\begin{definition}\label{cap-88}
  $A \subset X$. Then\footnote{Not abuse in general notation. This $\cpy$ differes fro
   \begin{align*}
    \cpy_\alpha(A) := \inf \cbr{ \cpy_\alpha(U) : U \supset A, \ U \text{ open.} }
   \end{align*}
\end{definition}
```

```
\begin{theorem}\label{cap-89}
  $\cpy_\alpha(A)$ is an {\bfseries (outer) Choquet capacity}, i.e.\footnote{Note (c)
   \begin{enumerate}[(a)]
     \item $A \subset B \Longrightarrow \cpy_\alpha(A) \leq \cpy_\alpha(B)$
    \item $A_n \uparrow A \Longrightarrow \cpy_\alpha(A_n) \uparrow \cpy_\alpha(A)$
    \item $K_n \downarrow K$ cpt sets $\Longrightarrow \cpy_\alpha(K_n) \downarrow \c
   \end{enumerate}
\end{theorem}
```

```
\begin{proof}
    \begin{enumerate}[(a)]
        \item $U, V$ open and $V \supset B \supset A$, so
            \begin{align*}
          \inf \cbr{ \cpy_\alpha(U) : U \supset A } \leq \inf \cbr{ \cpy_\alpha(V) : V \sups
            \end{align*}
            since there exists more $A$ covers.
        \item $A_n \uparrow A$. By definition $U_n \supset A_n$ open and
            \begin{align*}
          \cpy_\alpha(A_n) \leq \cpy_\alpha(U_n) \leq \cpy_\alpha(A_n) + \frac \epsilon{2
            \end{align*}
            If $U_n \uparrow$, we are done, since
            \begin{align*}
          \cpy_\alpha(A) \leq \cpy_\alpha(U) \os{\ref{cap-86}} \lim_n \cpy_\alpha(U_n) =
            \end{align*}
        But $U_n \longrightarrow U_1 \cup \dots \cup U_n \uparrow$, $U_1 \cup \dots \cup U_n
            \begin{align*}
          \cpy_\alpha(U_1 \cup \dots \cup U_n) \leq \cpy_\alpha(\underbracket[.5pt]{A_1 \
            \end{align*}
            Now $n \curvearrowright n+1$.
            \begin{align*}
          \cpy_\alpha(\underbracket[.5pt]{U_1 \cup \dots \cup U_n}_{U} \cup \underbracket
          &\os[\leq]{IA} \cpy_\alpha(A_n) + (1 - 2^{-n})\epsilon + \cpy_\alpha(A_{n+1}) +
            \end{align*}
            so ok.
        \item
            \begin{align*}
          \inf_n \cpy_\alpha(K_n) \os[\leq]{to show} \cpy_\alpha(K) \os[\leq]{\checkmark
            \end{align*}
            Let $\epsilon > 0 \ \exists U \supset K$ open.
            \begin{align*}
          \cpy_\alpha(K) \leq \cpy_\alpha(U) \leq \cpy_\alpha(K) + \epsilon.
            \end{align*}
        $\exists n_0 \ \forall n \geq n_0 : K_n \subset U$ (as $U \supset K = \bigcap_n K_n$)
            \begin{align*}
          \cpy_\alpha(K_n) \leq \cpy_\alpha(U) \leq \cpy_\alpha(K) + \epsilon
            \end{align*}
```

```
            make $\inf_n$, make $\epsilon \downarrow 0$.\qedhere
    \end{enumerate}
\end{proof}

We now use $\cpy_\alpha(A)$ as above.\\

{\bfseries Aim\hspace*{1em}} Explore \enquote{smallness}\\


{\bfseries Exercise}
    \begin{enumerate}[(1)]
        \item $\cpy_\alpha$ is strong sub-additive
        \item Using part (a), it holds $\forall A_n \subset B_n$
            \begin{align*}
           \cpy_\alpha(B_1 \cup \dots \cup B_n) - \cpy_\alpha(A_1 \cup \dots \cup A_n)
            \end{align*}
    \end{enumerate}
```

cap-810 **8.8 Definition.** (a) $\{F_n\}$ is **a(n $\mathcal{E}$-)nest**, if F_n is closed, $F_n \uparrow$ and $\lim_n \operatorname{cap}_\alpha(X \setminus F_n) = 0$.

(b) $\{F_n\}$ **regular ($\mathcal{E}$-)nest**, if

$$F_n = \operatorname{spt}(\mathbb{1}_{F_n} \cdot m).$$

$\iff \forall x \in F_n\ \forall$ neighbourhoods $U(x)$ of $x : m(F_n \cap U(x)) > 0$.

(c) $N \subset X$ is **($\mathcal{E}$-)exceptional**, if $\operatorname{cap}_\alpha(N) = 0 \iff \exists$ nest $\{F_n\} : N \subset \bigcap_n F_n^c$.

(d) A property $\Pi(x)$ holds **quasi-everywhere (q.e.)** if $\{x : \Pi(x)$ fails.$\}$ is exceptional.

(e) A q.e. defined $f : X \setminus$ exceptional $\to \overline{\mathbb{R}}$ is **quasi-continuous (q.c.)** if

$$\forall \varepsilon > 0\ \exists U = U_\varepsilon \subset X \text{ open, } \operatorname{cap}_\alpha(U) < \varepsilon : f\big|_{X \setminus U} \text{ is cts.}$$

Notation $\mathcal{C}\left(\{F_n\}\right) = \{u : u|_{F_n}$ cts, $\{F_n\}$ nest $, n = 1, 2, \dots\}$.

Exercise

(1) u is q.c. $\iff \exists$ nest and $u \in \mathcal{C}\left(\{F_n\}\right)$.

cap-811 **8.9 Lemma.** *$\{F_n\}$ nest. Then $\{F_n'\}$ with $F_n' := \operatorname{spt}(\mathbb{1}_{F_n} \cdot m)$ is a regular nest.*

Proof. Let F be closed, $F' := \operatorname{spt}(\mathbb{1}_F \cdot m)$. Then it is clear that $F' \subset F$ and $m(F \setminus F') = 0$ as F' is the smallest closed set s.t. $(F')^c$ is a $\mathbb{1}_F \cdot m$-null set). Now set $U' := X \setminus F'$, $U := X \setminus F$, then $m(U' \setminus U) = m(F' \setminus F) = 0$. Since $e_U, e_{U'}, \hat{e}_U, \hat{e}_{U'}$ are defined via $\mathcal{L}_U$, $\mathcal{L}_{U'}$ and as $\mathcal{L}_U = \mathcal{L}_{U'}$, we get $\operatorname{cap}_\alpha(U) = \operatorname{cap}_\alpha(U')$ (minimizers are equal). ■

cap-812 **8.10 Lemma.** *(a) $\mathcal{S}$ = countably many q.c. functions $\{f_k\}_k$. Then $\exists$ regular $\mathcal{E}$-nest $\{F_n\}_n$ with $\mathcal{S} \subset \mathcal{C}(\{F_n\})$ («uniform nest»).*

(b) $\{F_n\}$ is regular $\mathcal{E}$-nest, $u \in \mathcal{C}(\{F_n\})$, $u \geqslant 0$ m-a.s. $\Longrightarrow \forall x \in \bigcup_{n\in\mathbb{N}} F_n : u(x) \geqslant 0$ or, $u \geqslant 0$ q.e., respectively.

Proof. (a) $\forall k \exists$ nests $\{F_n^k\}_{n\in\mathbb{N}}$, $\operatorname{cap}_\alpha(X \setminus F_n^k) \leqslant \frac{1}{n2^k}$ and $f_k \in \mathcal{C}(\{F_n^k\}_n)$. Set

$$F_n := \bigcup_{k\in\mathbb{N}} F_n^k \quad \text{(closed)}.$$

Then

$$\operatorname{cap}_\alpha(X \setminus F_n) \overset{\text{-sub}}{\underset{\text{-additive}}{\leqslant}} \sum_k \underbrace{\operatorname{cap}_\alpha(X \setminus F_n^k)}_{\leqslant \frac{1}{n2^k}} \leqslant \frac{1}{n},$$

and $f_k \in \mathcal{C}(\{F_n\}_n)$.[5] By 8.9 $F_n' := \operatorname{spt}(\mathbb{1}_{F_n} \cdot m)$ does the trick.

(b) Assume $\exists n \exists x \in F_n : u(x) < 0$. $u\big|_{F_n}$ cts $\Longrightarrow \exists$ n'hood $U(x) : u\big|_{U(x)\cap F_n} < 0$. But, since we have a regular nest, $m(F_n \cap U(x)) > 0$. So this is a contradictionto $u \geqslant 0$ m-a.e.

■

cap-813 **8.11 Lemma** (Capacity finer than measure). $\operatorname{cap}_\alpha(A) = 0 \Longrightarrow m(A) = 0$.

Proof.

$$0 = \operatorname{cap}_\alpha(A) \overset{\text{def}}{=} \mathcal{E}_\alpha(e_A^\alpha, \underbrace{\hat{e}_A^\alpha}_{\in \mathcal{L}_A})$$

$$\overset{8.5\text{ (a)}}{\geqslant} \mathcal{E}_\alpha(e_A^\alpha) = 0$$

$\Longrightarrow \mathcal{E}_\alpha(e_A^\alpha) = 0$

$\overset{8.5\text{ (e)}}{\Longrightarrow} m(A) = 0.$ ■

Question For $\alpha, \beta > \gamma$. True? $\operatorname{cap}_\alpha \asymp \operatorname{cap}_\beta$.

[5] New nest is smaller than the «private nest» of f_k.

cap-814 **8.12 Definition.** $u, u_n : X \to \overline{\mathbb{R}}$ functions.

(a) $\tilde{u}$ is a **quasi-continuous (q.e.) modification** of u if

- $u = \tilde{u}$ m-a.e., i.e. $\tilde{u}$ is an m-version of u.[6]
- $\tilde{u}$ is q.c.

(b) $u_n \xrightarrow{n\uparrow\infty} u$ **q.e. uniformly** if

$$\forall \varepsilon \exists U = U_\varepsilon \text{ open: } \operatorname{cap}_\alpha(U) < \varepsilon \wedge u_n \xrightarrow[\text{uniformly on } X \setminus U]{n\uparrow\infty} u.$$

Exercise $\widetilde{u \pm v} \overset{\text{a.e.}}{\underset{\text{q.e.}}{=}} \tilde{u} \pm \tilde{v}$ (need joint nest, «$\Longrightarrow$ q.e.» b/o 8.9 b) we are q.c.).

Key to existence of $\tilde{u}$ is a Markov-type inequality.

cap-815 **8.13 Lemma.** $u \in \mathcal{F} \cap \mathcal{C}(X)$. *Then*

$$\operatorname{cap}_\alpha(|u| > \lambda) \leqslant \left(\frac{K_\alpha}{\lambda}\right)^2 \mathcal{E}_\alpha(u, u), \tag{8.6}$$ cap::eq0

where κ denotes the sector constant.

Proof. Set $U = \{|u| > \lambda\}$ open, $\frac{1}{\lambda}|u| \geqslant 1$ on $U \Longrightarrow \frac{1}{\lambda}|u| \in \mathcal{L}_U$. Then[7]

$$\begin{aligned}
\operatorname{cap}_\alpha(U) &\overset{\text{def}}{=} \mathcal{E}_\alpha(e_U^\alpha, \hat{e}_U^\alpha) \\
&\overset{8.5\text{ (a)}}{\leqslant} \mathcal{E}_\alpha\left(\left(\frac{1}{\lambda}|u|\right) \wedge 1, \hat{e}_U^\alpha\right) \\
&\overset{\text{sector}}{\leqslant} \frac{K_\alpha}{\lambda} \sqrt{\mathcal{E}_\alpha(|u| \wedge \lambda)}\sqrt{\mathcal{E}_\alpha(\hat{e}_U^\alpha)} \\
&\overset{3.17}{\leqslant} \frac{K_\alpha}{\lambda} \sqrt{\mathcal{E}_\alpha(u)}\sqrt{\mathcal{E}_\alpha(\hat{e}_U^\alpha)} \\
&\leqslant \frac{K_\alpha}{\lambda} \sqrt{\mathcal{E}_\alpha(u)}\sqrt{\mathcal{E}_\alpha(e_U^\alpha, \hat{e}_U^\alpha)},
\end{aligned}$$

then the claim follows by definition of $\operatorname{cap}_\alpha(U)$. Idea for the last inequality:

$$\mathcal{E}_\alpha(\hat{e}_U^\alpha) = \hat{\mathcal{E}}_\alpha(\hat{e}_U^\alpha) \leqslant \hat{\mathcal{E}}_\alpha(\hat{e}_U^\alpha, e_U^\alpha),$$

since $e_U^\alpha \in \mathcal{L}_U$, 8.5 (a). ■

[6] So keep in mind, $\tilde{u}$ is the function as a good candidate for the equivalence class u. So it is all about choice of a good L^2-representative.

[7] Use Theorem 3.17, $T(x) = |x| \wedge \lambda$ and show $|T(x) - T(y)| \leqslant |x - y|$.

Existence of $\tilde{u} \cong$ «Lusin theorem» for capacities.

cap-816 **8.14 Theorem.** *Let $(\mathcal{E}, \mathcal{F})$ regular SDF_γ. Then any $u \in \mathcal{F}$ has a q.c. modification $\tilde{u}$.*

Proof. [8] Fix $u \in \mathcal{F}$. By regularity $\exists (w_n)_n \subset F \cap \mathcal{C}_c(X) : w_n \xrightarrow[a.e.]{\mathcal{E}^s_\alpha} u$ (worse case take subsequence for a.e.). Then we find a subsequence $(u_n)_n \subset (w_n)_n$ with[9]

$$\mathcal{E}^s_\alpha(u_{n+1} - u_n) < 2^{-3n}.$$

Then by (8.6)

$$\operatorname{cap}_\alpha(|u_{n+1} - u_n| > 2^{-n}) \leqslant \frac{\kappa_\alpha^2 2^{2n}}{2^{3n}} = \frac{\kappa_\alpha^2}{2^n}.$$

Define

$$F_n = \bigcap_{k=n}^{\infty} \left\{ |u_{k+1} - u_k| \leqslant 2^{-k} \right\}, \quad U_n = X \setminus F_n.$$

Then the F_n are closed, $F_n \uparrow$. Now we get

$$\begin{aligned}\operatorname{cap}_\alpha(U_n) &= \operatorname{cap}_\alpha\left(\bigcup_{k=n}^{\infty} \left\{ |u_{k+1} - u_k| > 2^{-k} \right\}\right) \\ &\overset{(8.6)}{\leqslant} \sum_{k=n}^{\infty} \underbrace{\operatorname{cap}_\alpha\left(|u_{k+1} - u_k| > 2^{-k}\right)}_{\leqslant \frac{\kappa_\alpha^2}{2^k}} \leqslant 2\frac{\kappa_\alpha^2}{2^n}.\end{aligned}$$

Thus, $\{F_n\}$ are nests and by definition, $\forall N \in \mathbb{N}\ \forall x \in F_N\ \forall m, n > M \geqslant N$:

$$|u_n(x) - u_m(x)| \leqslant \sum_{k=M}^{\infty} |u_{k+1}(x) - u_k(x)| \leqslant 2 \cdot 2^{-M},$$

and so $u_n \xrightarrow[\text{on } F_N]{\text{uniformly}} \lim_{n\to\infty} u$. Set

$$\tilde{u}(x) := \begin{cases} \lim_{n\to\infty} u_n(x) & x \in \bigcup_{N\in\mathbb{N}} F_N \\ 0 & \text{else} \end{cases}$$

$\tilde{u}$ is cts on each F_N, $\{F_N\}$ is an $\mathcal{E}$-nest, i.e. $\tilde{u} \in \mathcal{C}(\{F_N\})$. So we need to show that $\tilde{u} \overset{\text{a.e.}}{=} u$.

$$\{\tilde{u} \neq u\} \subset \underbrace{\bigcap_N F_N^c}_{\text{cap null set}} \cup \underbrace{\left\{ u \neq \lim_{n\to\infty} u_n \right\}}_{m\text{-null set}}$$

■

[8] Note the analogueous to the proof of the Fischer-Riesz theorem.
[9] Note that we are on the diagonal so symmetric $\mathcal{E}$ or not does not matter.

Remark Nothing said about measurability. So we need to work with complete measure spaces.

cap-817 **8.15 Corollary** (Chebychev-type inequality). *$(\mathcal{E},\mathcal{F})$ as in 8.14, $u\in\mathcal{F}$. Then*[10]

$$\operatorname{cap}_\alpha(|\tilde u|>\lambda)\leqslant\frac{\kappa_\alpha^2}{\lambda^2}\mathcal{E}_\alpha(u). \tag{8.7}$$ cap::eq0

Proof. $u\in\mathcal{F}$, $\tilde u, u_n$ as in proof of 8.14. Then by 8.14

$$\forall\varepsilon>0\ \exists U=U_\varepsilon\text{open: }\operatorname{cap}_\alpha(U_\varepsilon)<\varepsilon : u_n\xrightarrow[\text{on } X\setminus U_\varepsilon]{\text{uniformly}}\tilde u.$$

Now

$$\{|\tilde u|>\lambda\}\overset{\forall n\geqslant N_\varepsilon}{\subset}\left\{\left|u_n\right|>\lambda-\varepsilon\right\}\ \cup U_\varepsilon$$

Then

$$\begin{aligned}\operatorname{cap}_\alpha(|\tilde u|>\lambda)&\overset{8.7\text{ (b)}}{\leqslant}\underbrace{\operatorname{cap}_\alpha(\left|u_n\right|>\lambda-\varepsilon)}_{\leqslant\frac{\kappa_\alpha^2}{(\lambda-\varepsilon)^2}\mathcal{E}_\alpha(u_n)}+\underbrace{\operatorname{cap}_\alpha(U_\varepsilon)}_{\leqslant\varepsilon}\\&\xrightarrow[\varepsilon\text{ fixed}]{n\to\infty}\frac{\kappa_\alpha^2}{(\lambda-\varepsilon)^2}\mathcal{E}_\alpha(u)+\varepsilon\quad(u_n\xrightarrow{\mathcal{E}_\alpha^s}u)\\&\xrightarrow{\varepsilon\downarrow 0}\frac{\kappa_\alpha^2}{\lambda^2}\mathcal{E}_\alpha(u).\end{aligned}$$

■

Remark In general, we have $\operatorname{cap}_\alpha(A)=0\Longrightarrow m(A)=0$
Have $\beta\geqslant\alpha>\gamma$: $\operatorname{cap}_\alpha(A)\leqslant\text{const.}_{\kappa_\alpha^6}\operatorname{cap}_\beta(A)$

cap-818 **8.16 Corollary.** *$(\mathcal{E},\mathcal{F})$ regular SDF_γ, $(u_n)\subset\mathcal{F}$, $\mathcal{E}_\alpha^s$-Cauchy $(\alpha>\gamma)$, $\tilde u_n$ q.c.-modifications $\exists(\tilde u_{n(k)})_k\subset(\tilde u_n)_n$ $\exists$ q.c. $\tilde u\in\mathcal{F}$ s.t. $\tilde u_{n(k)}\xrightarrow[\text{and in }\mathcal{E}_\alpha^s]{\text{q.e. uniformly}}\tilde u$.*

Proof. Use (8.7) and 8.13 and get[11]

$$\operatorname{cap}_\alpha\left(\left|\tilde u_{n(k+1)}-\tilde u_{n(k)}\right|>2^{-k}\right)\leqslant 2^{-k}$$

[10] Note $\mathcal{E}_\alpha(u)=\mathcal{E}_\alpha(\tilde u)$, $\tilde u$ is L^2-representative of u.

[11] Mind: $\tilde u_{n(k)}$ is a suitable subsequence s.t. $\mathcal{E}_\alpha(\tilde u_{n(k+1)}-\tilde u_{n(k)})\leqslant 2^{-k}$ and $\mathcal{E}_\alpha(\tilde w)=\mathcal{E}_\alpha(w)$, $u\tilde{\pm}v=\tilde u\pm\tilde v$. Choose the *same nest for everything, ok by 8.10*.

Now take a joint nest $(\Phi_n)_n$ of $(\tilde{u}_n)_n$

$$F_k = \bigcap_{l \geqslant k} \left\{ \left| \tilde{u}_{n(l+1)} - \tilde{u}_{n(l)} \right| \leqslant 2^{-l} \right\} \cap \Phi_{n(l)}$$

are closed. Use $U_k = X \setminus F_k$ open.

Now $(F_k)_{k \in \mathbb{N}}$ is a nest

Set $\tilde{u}(x) := \begin{cases} \lim_{k\to\infty} \tilde{u}_{n(k)}(x) & \text{on } \bigcup_k F_k \\ 0 & \text{else} \end{cases}$

$\Longrightarrow \tilde{u} \in \mathcal{C}(\{F_k\})$ and so $u = \tilde{u}$ m-a.e. ■

Now let $(\mathbf{G}_\alpha)_{\alpha>\gamma}$, $(T_t)_{t\geqslant 0}$ be resolvent and semigroup given by $(\mathcal{E}, \mathcal{F})$ (→ § 3.1) on $L^2(m)$.

But Lemma 3.13 Extend $\mathbf{G}_\alpha$, T_t onto $L^\infty(m)$

Mind $T_t, \mathbf{G}_\alpha : L^\infty(m) \to L^\infty(m)$, $\mathcal{B}_b(E)$ = bdd Borel functions $\subset L^\infty(m) \to L^\infty(m)$.

Mind Original on L^2 and extension on L^∞ get the same notation.

cap-819 **8.17 Proposition.** $(\mathcal{E}, \mathcal{F})$ *regular* SDF_γ, *resolvent* $(\mathbf{G}_\alpha)_{\alpha>\gamma}$. *Then* $\forall \alpha > \gamma \; \forall f \in L^\infty(m)$ *s.t.* $\{|f| \geqslant \varepsilon\}$ *cpt for any* $\varepsilon > 0$, *we have:* $\mathbf{G}_\alpha f$ *has q.c. modification.*

Proof. WLOG (b/o linearity) $f \geqslant 0$, $\beta \geqslant \alpha > \gamma$.

1° Assume $f \in L^2_+(m) \cap L^\infty(m)$. Then $\mathbf{G}_\beta f \in \mathcal{F}$ and $\widetilde{\mathbf{G}_\beta f}$ exists and

$$\left\| \widetilde{\mathbf{G}_\beta f} \right\|_{L^\infty(m)} = \left\| \mathbf{G}_\beta f \right\|_{L^\infty(m)} \leqslant \frac{1}{\beta} \|f\|_{L^\infty(m)},$$

ok, $\frac{1}{\beta-\gamma}$ in $L^2(m)$.

2° m Radon, X topologically good, $\{f \geqslant \varepsilon\}$ cpt. $\Longrightarrow \exists$ cpt $K_n \uparrow$, $m(K_n) < \infty$ and $f_n := f \mathbb{1}_{K_n} \in L^\infty(m) \cap L^2(m)$ and $f_n \xrightarrow{\text{uniformly}} f$.

3° Take a joint nest for all objects (use 8.10), then

$$\left\| \widetilde{\mathbf{G}_\beta f_n} - \widetilde{\mathbf{G}_\beta f_k} \right\|_{L^\infty(m)} = \left\| \widetilde{\mathbf{G}_\beta (f_n - f_k)} \right\|_{L^\infty(m)} \leqslant \frac{1}{\beta} \left\| f_n - f_k \right\|_{L^\infty(m)} \xrightarrow{n,k\to\infty} 0.$$

So $\exists (R_\beta f) : \widetilde{\mathbf{G}_\beta f_n} \xrightarrow{\text{uniformly}} R_\beta f$, hence, $R_\beta f$ q.c. and

$$R_\beta f = \lim_{n\to\infty} \widetilde{\mathbf{G}_\beta f_n} \overset{\text{a.e.}}{=} \lim_{n\to\infty} \mathbf{G}_\beta f_n \overset{\text{a.e.}}{=} \mathbf{G}_\beta f$$ ■

cap-820 **8.18 Remark.** ! 8.17 does not give an operator R_β but a function $R_\beta f$, means, nest depends on f.

Want q.c. modification of $T_t f$.

Strategy $T_t\left(L^2(m)\right) \overset{!!}{\subset} \mathfrak{D}(\mathbf{A}) \underset{\text{dense}}{\overset{3.6}{\subset}} \mathcal{F}$, $\in$ means T_t is improving regularity.

cap-821 **8.19 Theorem** ([RS75], Theorem X.52). *(a)* $(T_t)_{t\geqslant 0}$ *contraction semigroup on* $L^2(m)$ *which is analytic, i.e.*

$$z \to T_z u,\ z \in S(\tilde\kappa) = \{|\Im z| < \tilde\kappa\, \Re z\} \tag{8.8}$$ cap::eq0

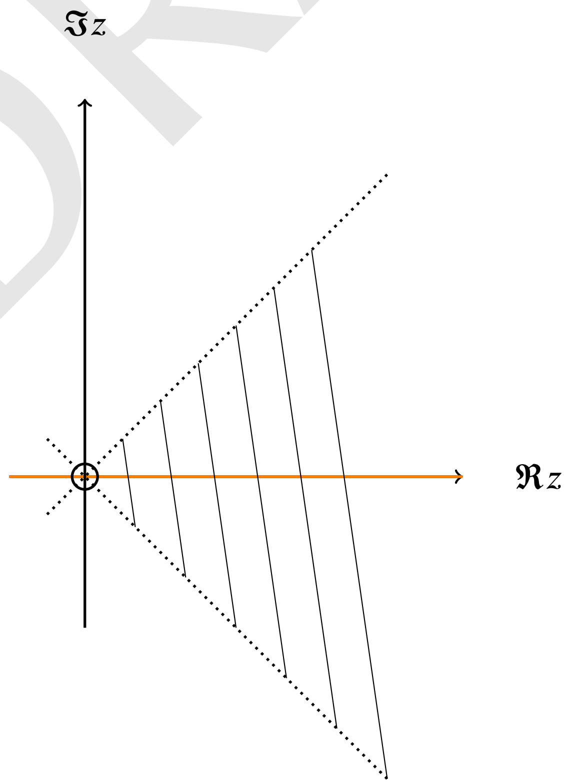
is a extension, it is analytic, $u \in L^2(m)$*, it is a semigroup on* $S(\tilde\kappa)$*. Then*

$$T_t\left(L^2(m)\right) \subset \mathfrak{D}(\mathbf{A}).$$ [12]

(b) $(\mathcal{E}, \mathcal{F})$ SDF_γ. *Then the sector condition*

$$|\mathcal{E}_\alpha(u,v)| \leqslant \kappa_\alpha \sqrt{\mathcal{E}_\alpha(u)}\sqrt{\mathcal{E}_\alpha(v)}$$

$\Longrightarrow \left(e^{-\alpha t} T_t\right)_{t\geqslant 0}$ *analytic.*



Recall $\mathcal{E}_\alpha(u,v) \overset{\text{Exercise, use}}{\underset{3.5\text{ (b)}}{=}} \langle \alpha u - \mathbf{A}u, v\rangle_{L^2(m)}$ if $u \in \mathfrak{D}(\mathbf{A})$
$\Longrightarrow$ sector condition is a condition on $(\alpha - \mathbf{A})$
$\Longrightarrow$ use «soft» analysis.

cap-822 **8.20 Corollary.** *Let $(T_t)_{t\geqslant 0}$ be a semigroup given by SDF_γ $(\mathcal{E},\mathfrak{F})$. Then $T_t f$ has for all $f \in L^\infty(m)$ s.t. $\{|f| \geqslant \varepsilon\} \subset$ cpt. set $\forall \varepsilon > 0$. Then $t_t f$ has a quasi-continuous modification $\tilde{T}_t f$.*

Proof. Mimic the proof of 8.17. ■

Chapter 9

Markov processes

Setting

- $(E, \mathcal{B}(E))$ measurable (topological) space
- $(\Omega, \mathcal{A}, \mathbb{P})$ probability space
- $E_\Delta := E \cup \{\Delta\}$ **cemetry** or **coffin state**
- E compact, Δ is a new isolated point, else: 1-point compactification
- Filtration, i.e. $\mathcal{A}_t \subset \mathcal{A}$ σ-algebras. $s \leqslant t : \mathcal{A}_s \subset \mathcal{A}_t$, $\mathcal{A}_\infty = \sigma\left(X_s : 0 \leqslant s < \infty\right)$

random variable $\quad X : (\Omega, \mathcal{A}) \xrightarrow{\text{measurable}} (E, \mathcal{B})$

stochastic process $\quad (\Omega, \mathcal{A}, \mathbb{P}, X_t, t \geqslant 0, E)$

Markov process $\quad (\Omega, \mathcal{A}, \mathbb{P}^x, x \in E, X_t, t \geqslant 0, E)$

(M1)
- $X_t(\omega) = \Delta$ for all $t \geqslant \xi(\omega) = \inf\left\{s \geqslant 0 : X_s(\omega) = \Delta\right\} \in [0, \infty]$[1]
- $\forall t > 0\ \exists \vartheta_t : \Omega \to \Omega : X_s(\vartheta_t \omega) = X_{s+t}(\omega)$, the shift.
- $t \overset{\text{path}}{\mapsto} X_t(\omega)$ is càdlàg (right-cts in $[0, \infty)$, left limits in $(0, \infty)$) $\forall \omega$

(M2) $x \mapsto \mathbb{P}^x(X_t \in B)$ measurable $\forall t\ \forall B \in \mathcal{B}(E)$ the transition probability

(M3) $\forall s, t\ \omega \mapsto X_t(w)$ $\mathcal{A}_t$-measurable, i.e. «adapted», the filtration is right-continuous

$$\mathcal{A}_t = \mathcal{A}_{t+} \overset{\text{def}}{=} \bigcup_{\varepsilon > 0} \mathcal{A}_{t+\varepsilon}$$

and the Markov property holds

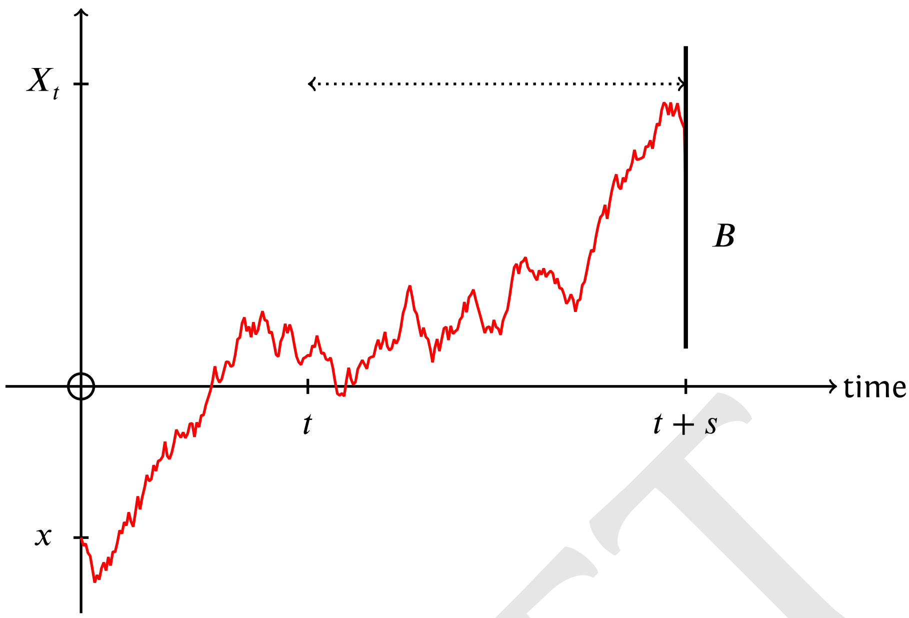
$$\forall B \in \mathcal{B}(E) : \mathbb{P}^x(X_{t+s} \in B \mid \mathcal{A}_t) = \mathbb{P}^{X_t}(X_s \in B)$$

$$\overset{\text{mind}}{=} \mathbb{P}^x(X_t \in \mathrm{d}y) \text{ a.s.}$$

(M4) $\mathbb{P}^\Delta(X_t = \Delta) = 1\ \forall t \geqslant 0$

[1] Says: once I in the grave, i stay there.

(M5) $\mathbb{P}^x(X_0 = 0) = 1 \ \forall x \in E$



$$\mathbb{P}^{X_t}(X_s \in dy) = \mathbb{P}^x(X_{t+s} \in dy \mid A_t) = \mathbb{P}^x(X_{t+s} \in dy \mid X_t)$$

rationale one-step transitions $\mathbb{P}^x(X_t \in dy)$ are all we need to know!

- **Starting distributions** μ = probability $(E_\Delta, \mathcal{B}(E_\Delta))$
- $\Gamma \in \mathcal{A}_\infty : \mathbb{P}^\mu(\Gamma) \overset{\text{def}}{=} \int_{E_\Delta} \mathbb{P}^x(\Gamma)\mu(dx) \ (X_0 \sim \mu, \mathbb{P}^x = \mathbb{P}^{\delta_x})$
- $\mathcal{F}_t^\nu$:= completion of $\mathcal{A}_t$ w.r.t ν
- $\mathcal{F}_t := \bigcup_\nu \mathcal{F}_t^\nu$
- **Stopping time** $\sigma : \Omega \to [0, \infty] : \{\sigma \leqslant t\} \in \mathcal{A}_t \forall t$
- $\mathcal{A}_\sigma := \{\Gamma \in \mathcal{A}_\infty : \Gamma \cap \{\sigma \leqslant t\} \in \mathcal{A}_t, \ \forall t \geqslant 0\}$
- **Strong Markov process** is a Markov process $\oplus$

  $\forall B \in \mathcal{B}(E) : \mathbb{P}^\mu(X_{t+\sigma} \in B \mid A_\sigma) = \mathbb{P}^{X_\sigma}(X_t \in B)$ a.s. w.r.t $\mathbb{P}^\mu(X_\sigma \in dy)$ on $\{\sigma < \infty\}$

- **quasi-left-continuous (qlc)** $\sigma_n, n \in \mathbb{N}$, stopping times, $\sigma_n \uparrow \sigma$ stopping time

$$\mathbb{P}^\mu\left(\lim_{n\to\infty} X_{\sigma_n} = X_\sigma, \ \sigma < \infty\right) = \mathbb{P}^\mu(\sigma < \infty)$$

Example $\sigma = s$, $\sigma_n = s - \frac{1}{n} \uparrow \sigma$. Then by qlc

$$X_{s-\frac{1}{n}} \xrightarrow{a.s.} X_s \text{\#\#\#} s \mapsto X_s \text{left-cts}$$

The problem is that the «a.s.» has an exceptional null set depending on $\left(s - \frac{1}{n}\right)_{n\in\mathbb{N}}$

A **Hunt process** is a quasi-left-continuous strong Markov process.

How to construct a Markov process Assume $(X_t)_{t\geqslant 0}$ is a Markov process.

$$P_t f(x) := \mathbb{E}^x f(X_t), \quad f \in \mathcal{B}_b(E),\ t \geqslant 0 \tag{9.1}$$ mp::eq01

Claim P_t is a sub-Markovian semigroup

(a) $x \mapsto P_t f(x)$ measurable, i.e. $P_t : \mathcal{B}_b(E) \to \mathcal{B}_b(E)$
Indeed Standard «Sombrero-Lemma» argument $x \mapsto \mathbb{P}^x(X_t \in B) = P_t \mathbb{1}_B(x)$ is measurable resp. $\int f(y) \underbrace{\mathbb{P}^x(X_t \in dy)}_{\text{measurable in } x}$, $f = \lim_n(\text{ step fns })_n$

(b) $P_{t+s} f(x) \overset{\text{def}}{=} \mathbb{E}(\mathbb{E}^x(f(X_{t+s} | \mathcal{A}_t)) = \mathbb{E}^x(\mathbb{E}^{X_t}(f(X_s)) = \mathbb{E}^x P_s f(X_t) = P_t(P_s f)(x)$

(c) P_t is sub-Markov: $0 \leqslant f \leqslant 1 \Longrightarrow 0 \leqslant \mathbb{E}^x f(X_t) \leqslant 1$.[2]

Other «semigroup» properties need more assumptions, e.g. (C_0) ($\longleftrightarrow t \underset{\text{in } \mathbb{P}^x}{\overset{\text{cts}}{\mapsto}} X_t$) or $P_t : C_b \to C_b$ (Feller property)

Main point Markov property yields a construction principle of a Markov process given P_t, $\forall f, g \in \mathcal{B}_b(E)$, $\forall s, t \geqslant 0$, $s < t$

$$\begin{aligned}
\mathbb{E}^x f(X_s) g(X_t) &= \mathbb{E}^x(\mathbb{E}^x(f(X_s) g(X_t) \mid \mathcal{A}_s)) \\
&\overset{\text{MP}}{=} \mathbb{E}^x(f(X_s)\, \mathbb{E}^{X_s} g(X_{t-s})) \\
&= \mathbb{E}^x(f(X_s) P_{t-s} g(X_s)) \\
&= P_s(f \cdot P_{t-s} g)(x),
\end{aligned}$$

e.g. $f = \mathbb{1}_A, g = \mathbb{1}_B \longrightarrow \mathbb{P}^x(X_s \in A, X_t \in B)$. So, by iteration,

[2] Note that $\mathbb{E}^x \mathbb{1}_{E_\Delta} = 1$, $\mathbb{E}^x \mathbb{1}_E$ can be < 1. Mass can vanish to ∞. So $f \in \mathbb{B}(E)$ always means $f(\Delta) := 0$

mp-91 **9.1 Lemma.** *Let* $(X_t)_{t\geqslant 0}$ *be a Markov process.* $0 \leqslant t_0 < t_1 < \cdots < t_n$, $n \in \mathbb{N}$, $B_1, \dots, B_n \in \mathcal{B}(E)$, $\mu = \delta_x$. *Then*

$$\int \mathbb{P}^x(X_{t_i} \in B_i,\ i = 1, \dots, n)\mu(\mathrm{d}x) = \int P_{t_1}\left[\mathbb{1}_{B_1} P_{t_2-t_1}\left[\mathbb{1}_{B_2} P_{t_3-t_2}\left[\dots\left[P_{t_n-t_{n-1}}\mathbb{1}_{B_n}\right]\right]\right]\right](x)\mu(\mathrm{d}x) \tag{9.2}$$

mp::eq02

Basic result on process construction.

mp-92 **9.2 Theorem** (Kolmogorov, 1933). *Let* $p_{t_1}, \dots, p_{t_n}$, $0 \leqslant t_1 < \dots < t_n$ *be consistent (projective) probabilities on* $\mathcal{B}(E)$, *i.e.*

- $p_{t_1,\dots,t_n}(B_1 \times \dots \times B_n) = p_{t_{\sigma(1)},\dots,t_{\sigma(n)}}(B_{\sigma(1)} \times \dots B_{\sigma(n)})$ $\forall$ *permutations* $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$
- $p_{t_1,\dots,t_{n-1},t_n}(B_1 \times \dots \times B_{n-1} \times E) = p_{t_1,\dots,t_{n-1}}(B_1 \times \dots \times B_{n-1})$.

Then there exists a stochastic process $(X_t)_{t\geqslant 0}$ *such that*

$$p_{t_1,\dots,t_n}(B_1 \times \dots \times B_n) = \mathbb{P}(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n) \tag{9.3}$$

Note $(X_t)_{t\geqslant 0}$ is MP $\iff$ $p_{t_1,\dots,t_n}$ have structure of Lemma **??** ($\Longrightarrow$ ✓, $\Longleftarrow$ extra work)

Chapter 10

From regular SDF$_\gamma$ to Hunt process

9.1 $\oplus$ 9.2 $\Longrightarrow$ MP can be built from P_t or more precisely

$$p_t(x,B) := \mathbb{P}^x(X_t \in B) = P_t \mathbb{1}_B(x) \overset{\text{a.e.}}{=} \underbrace{\boxed{T_t \mathbb{1}_B(x)}}_{\in\, L^2(X),\ m(B)<\infty}$$

is a measurable function in x and (T_t) the semigroup given by $(\mathcal{E}, \mathcal{F})$. But

$$p_t(x,B) := T_t \mathbb{1}_B(x)$$

is not good, depends on an exceptional null set $N = N(t,B)$.

Work Clever choice of representative

A **(Measure/Markov) kernel** $p_t(x,B)$ $t \geqslant 0, x \in X, B \in \mathcal{B}(X)$

- $x \mapsto p_t(x,B)$ measurable $\forall t \geqslant 0$ $\forall B \subset X$ Borel
- $B \mapsto p_t(x,B)$ **sub-probability measure** $\forall t \geqslant 0$ $\forall x \in X$, i.e. $p_t(x,X) \leqslant 1$
- the Chapman-Kolmogorov equation holds $\forall s \leqslant t$ $\forall x \in X \forall B \in \mathcal{B}(X)$

$$p_{t+s}(x,B) = \int p_t(x,\mathrm{d}y) p_s(y,B)$$

$\Longleftrightarrow$ semigroup property $\Longleftrightarrow$ Markov property of process

Notation $p_t f(x) := \int f(y) p_t(x,\mathrm{d}y)$, «identity the semigroup with its representing kernel»[1]

Idea for rep. Take $T_t \tilde{\mathbb{1}}_B$ = q.c. modification = a single function

Need «$\tilde{T}_t \mathbb{1}_B$» need to modify the semigroup.

[1] Exercise: b/o Chapman-Kolmogorov

Problem, losing q.c. modification $\Longrightarrow$ get nests $\{F_k^0\}_k$, $E := \bigcup_{k\in\mathbb{N}} F_k^0$, $\mathrm{cap}_\alpha(X \setminus E) = 0$, can take (by 8.10) *one* nest for all objects ($\Longrightarrow$ need countability)

Drawback New state space E, «loss» $X \setminus E$ non-constructive[2]

Setting

- $(\mathcal{E}, \mathcal{F})$ = regular SDF_γ
- $(\mathbf{G}_\alpha)_{\alpha>\gamma}$ (L^2-)resolvent, $(T_t)_{t\geqslant 0}$ (L^2-)semigroup

Know $\mathbf{G}_\alpha$, T_t «live» also on $L^\infty(m)$ (harparrowright cf. §3, §4)

hp-101 **10.1 Lemma.**

$$\mathbf{G}_\alpha f := \sum_{n=0}^{\infty} (\beta-\alpha)^n \mathbf{G}_\beta^{n+1} f, \quad f \in L^\infty(m),\ 0 < \alpha \leqslant \gamma < \beta \tag{10.1}$$ hp::eq01

extend $(\mathbf{G}_\alpha)_{\alpha>\gamma}$ *to all* $\alpha > 0$ *(in* $L^\infty(m)$*).*

Proof. $\mathbf{G}_\beta$ is sub-Markov. Then by (**??**)

$$\left\|\mathbf{G}_\beta f\right\|_{L^\infty(m)} \leqslant \frac{1}{\beta} \|f\|_{L^\infty(m)}$$

$$\left\|\mathbf{G}_\beta f\right\|_{L^2(m)} \leqslant \frac{1}{\gamma-\beta} \|f\|_{L^2(m)}$$

Hence

$$\left\|\sum_0^\infty (\beta-\alpha)^n \mathbf{G}_\beta^{n+1} f\right\|_{L^\infty(m)} \leqslant \sum_0^\infty (\beta-\alpha)^n \left\|\mathbf{G}_\beta^{n+1} f\right\|_{L^\infty(m)}$$

$$\sum_0^\infty (\beta-\alpha)^n \frac{1}{\beta^{n+1}} \|f\|_{L^\infty(m)} = \underbrace{\frac{1}{\beta}\frac{1}{1-\frac{\beta-\alpha}{\beta}}}_{=\frac{1}{\alpha}} \|f\|_{L^\infty(m)}$$

So (10.1) makes sense. Why extension? By the resolvent identity

$$\mathbf{G}_\alpha f = \mathbf{G}_\beta f + (\beta-\alpha)\mathbf{G}_\beta \mathbf{G}_\alpha f, \quad f \in L^\infty(m) \cap L^2(m),\ \alpha, \beta > \gamma$$

and by iteration, we get

$$\mathbf{G}_\alpha f = \mathbf{G}_\beta f + (\beta-\alpha)\mathbf{G}_\beta^2 f + (\beta-\alpha)^2 \mathbf{G}_\beta^2 \mathbf{G}_\alpha f.$$

Now it is easy to see that (10.1) follows. So (10.1) gives indeed an extension satisfying the resolvent identity. ■

[2] So we have a process which cannot start everywhere.

The next lemma addresses our countability issue.

hp-102 **10.2 Lemma** (Fukushima). *$\mathbb{F}$ any set, $\mathbb{B} \subset \mathbb{F}$ countable, Σ countable many functions such that $\mathbb{F} \times \mathbb{F} to \mathbb{F}$. Then there is a countable set $\mathbb{D}$ with*

(a) $\mathbb{B} \subset \mathbb{D} \subset \mathbb{F}$

(b) $\forall s \in \Sigma : s(\mathbb{D} \times \mathbb{D}) \subset \mathbb{D}$

Proof. Let $(s_1, s_2, s_3, \dots)$ be q sequence s.t. every $s \in \Sigma$ appears ∞ often in $(s_1, s_2, s_3, \dots)$ (e.g. $\Sigma \times \Sigma$ and enumerate it if Σ is infinite). Now define

$$\begin{aligned} \mathbb{B}_1 &:= \mathbb{B} \\ \mathbb{B}_{n+1} &:= \mathbb{B}_n \cup s_{n+1}(\mathbb{B} \times \mathbb{B}_n), \quad n = 1, 2, \dots \\ \mathbb{D} &= \bigcup_{n=1}^{\infty} \mathbb{B}_n \end{aligned}$$

and

$$\forall f, g \in \mathbb{D}\ \forall s \in \Sigma\ \exists n : f, g \in \mathbb{B}_n : s = s_{n+1}.$$

■

(I) Construct kernels $\tilde{p}_t$, $t \in \mathbb{Q}_+$, $t \geqslant 0$

hp-103 **10.3 Lemma.** *There is a regular nest $(F_k^0)_k$ and Markov kernels $(\tilde{p}_t(\cdot, \cdot))_{t \in \mathbb{Q}_+}$ and $\tilde{R}_\alpha$ ($\mathbb{Q}_+ \ni \alpha > \gamma$) with*

(a) $\tilde{p}_t(C_\infty(X)) \subset \mathfrak{C}_\infty\left(\left\{F_k^0\right\}\right)$, $t \in \mathbb{Q}_+$[3] *and* $\tilde{R}_\alpha(\mathfrak{C}_\infty(X)) \subset \mathfrak{C}_\infty\left(\left\{F_k^0\right\}\right)$, $\alpha > \gamma$, $\alpha \in \mathbb{Q}_+$

(b) $\tilde{p}_t u = \tilde{T_t} u$, $\tilde{R}_\alpha u = \tilde{\mathbf{G}_\alpha} u\ \forall u \in L^2(m), t \in \mathbb{Q}_+, \mathbb{Q}_+ \ni \alpha > \gamma$

Proof. 1° **Claim** $\exists$ countable $B_0 \subset \mathfrak{F} \cap C_c(X)$

- B_0 dense in $C_c(X)$
- $u, v \in B_0$, $a \in \mathbb{Q}$[4]

[3]Note that $\mathfrak{C}_\infty(X) = \overline{C_c(X)}^{\|\cdot\|_\infty} = \left\{u \in C(X) : \forall \varepsilon > 0\ \exists K_\varepsilon \text{ cpt: } \left\|u \mathbb{1}_{K_\varepsilon^c}\right\|_\infty \leqslant \varepsilon\right\}$.

[4]Note that $x \wedge y = \frac{1}{2}(x + y - |x - y|)$ and $x \vee y = \frac{1}{2}(x + y + |x - y|)$. So $|u| \iff u \wedge v \in B_0$ or $u \vee v \in B_0$.

Indeed Take $(u_k)_{k\in\mathbb{N}} \in C_c(X)$ dense.

B/o regularity $\forall k,n\ \exists u_{k,n} \in \mathcal{F}\cap C_c(X) : \left\|u_k - u_{k,n}\right\|_\infty \leqslant \frac{1}{n}$.

Clear (triangle inequality) $\mathcal{G} := \left\{u_{k,n} : k,n \in \mathbb{N}\right\} \subset C_c(X)$ dense

Consider

$$\begin{aligned} s_0(u,v) &:= |u| \\ s_1(u,v) &:= u+v \\ s_{2,a}(u,v) &:= a\cdot u, a\in\mathbb{Q} \end{aligned}$$

with $\left(\mathcal{F}\cap C_c(X)\right)^2 \to \mathcal{F}\cap C_c(X)$. Lemma 10.2 applied gives set B_0.

2° $H_0 := \bigcup_{t\in\mathbb{Q}_+,\mathbb{Q}_+\ni\alpha>\gamma} \left(T_t(B_0)\cup \mathbf{G}_\alpha(B_0)\right) \subset \mathcal{F}$, since T_t is analytic and by 2.11 $\mathbf{G}_\alpha(\mathcal{F}) \subset \mathcal{D}(\mathbf{A}) \subset \mathcal{F}$ the «⊂» follows from 8.19. Then H_0 is countable and by Lemma 8.10: ∃ one (!) common nest $\left\{F_k^0\right\}$ with

$$\tilde{H}_0 := \left\{\tilde{u} : \tilde{u}\in H_0\right\} \subset \mathcal{C}_\infty(\left\{F_k^0\right\})$$

3° Set $E := \bigcup_{k\in\mathbb{N}} F_k^0 \subset X$, $\operatorname{cap}_\alpha(X\setminus E) = 0$. Use 8.20 (and 8.10 to get joint nests!). So $\forall x\in E$, $u,v\in B_0$, $t\in\mathbb{Q}_+$, $a\in\mathbb{Q}$[5]

$$\begin{aligned} \widetilde{T_t(u+v)}(x) &= \tilde{T_t}u(x) + \tilde{T_t}v(x) \\ \widetilde{T_t(au)}(x) &= a\tilde{T_t}u(x) \\ 0 \leqslant \tilde{T_t}u(x) &\leqslant 1 \end{aligned}$$

$\Longrightarrow \left|\tilde{T_t}u(x)\right| \leqslant \|u\|_{L^\infty(m)} = \|u\|_\infty$ (spt $m = X$, full support).

$u\in B_0, u\geqslant 0 \Longrightarrow M := \|u\|_\infty < \infty$, take $a\in\mathbb{Q}_+$, $a > M$.
$\Longrightarrow \frac{u}{a}\in B_0, 0\leqslant \frac{u}{a}\leqslant 1, 0\leqslant \tilde{T_t}\frac{u}{a}(x) = \frac{1}{a}\tilde{T_t}u(x)\leqslant 1$ multiply ba a, let $a\downarrow \|u\|_\infty$ along $\mathbb{Q}_+$.

4° $x\in E$, Define $l_x(u) := \tilde{T_t}u(x)$, $u\in B_0$

Claim $u\mapsto l_x(u)$ positive, bounded, linear (✓ 3°) and extends to $C_\infty(X)$ by continuity

[5]Last line: $0\leqslant u\leqslant 1$ a.e., argument: $0\leqslant T_t u\leqslant 1$ a.e. (Markov) $\longrightarrow 0\leqslant \tilde{T_t}u\leqslant 1$ a.e. ($\tilde{T_t}u = T_t u$ a.e.) $\longrightarrow$ *q.e.* (8.12 b), maybe make except set lower -- it is «uindexed» by $t\in\mathbb{Q}_+$, $u\in B_0$.

Indeed $\forall u \in C_\infty(C)\ \exists (u_n)_{n\in\mathbb{N}} \subset B_0 : \|u_n - u\|_\infty \xrightarrow{n\uparrow\infty} 0$ ($B_0 \overset{\text{dense}}{\subset} C_c \overset{\text{dense}}{\subset} C_\infty$). Then[6]

$$l_x(u) := \lim_{n\to\infty} \tilde{T_t} u(x) \text{ exists uniformly}$$
$$l_\bullet(u) \in \mathcal{C}_\infty(\{F_k^0\}),\ \text{i.e. } \tilde{T_t} u_n \text{ cts on } F_k^0\ \forall k$$

5° By Riesz representation theorem ($x \in E$ fixed)

$$l_x(u) = \int u(y)\tilde{p}_t(x, \mathrm{d}y) \overset{\text{def}}{=} \tilde{p}_t u(x) \quad \forall u \in C_\infty(X),$$

where $\tilde{p}_t(x, \mathrm{d}y)$ is a unique, regular Radon measure ($x \in E$).

- $\tilde{p}_t$ are sub-probability measures,

$$\tilde{p}_t(x, X) \underset{u_n \in C_c^+(X),\, u_n \uparrow 1}{\overset{\text{BL}}{=}} \sup_n \int u_n(y)\tilde{p}_t(x, \mathrm{d}y) = l_x(u_n)$$
$$\leqslant \sup_n \|u_n\|_\infty \leqslant 1$$

- $U \subset X$ open, $\exists u_n \in C_c(X)$, $u_n \uparrow \mathbb{1}_U$ (Urysohn). As above it follows

$$\tilde{p}_t(x, U) = \sup_n l_x(u_n) \overset{\text{def}}{=} \sup_n \tilde{T_t} u_n(x)$$

$\implies x \mapsto \tilde{p}_t(x, U)$ measurable $\forall U$ open.

By regularity[7]

$$\tilde{p}_t(x, B) = \inf_{U \supset B \text{ open}} \tilde{p}_t(x, U),$$

WLOG inf over countably many open sets (X is separable!). So $x \mapsto \tilde{p}_t(x, B)$ is measurable.

If $x \notin E$ $\quad \tilde{p}_t(x, B) = 0$. Measurability preserved as $m(X \setminus E) = 0$ and m complete.
$\implies \tilde{p}_t(x, B)$ ($\forall x \in X\ \forall B \in \mathcal{B}(X)$) a kernel.

---

[6] $\tilde{T_t} u_n - \tilde{T_t} u_m \overset{3^\circ}{=} \widetilde{T_t(u_n - u_m)} \overset{3^\circ}{\leqslant} \|u_n - u_m\|_\infty$.
[7] One could also use a monoton class argument.

6° L^2-extension. $\forall u \in C_c(X)$

$$\tilde{p}_t u \overset{\text{a.e.}}{=} \tilde{T_t u},\ \text{WLOG } u \geqslant 0 \text{ (else } u^\pm \text{ etc.)}$$
$$\exists w \in B_0,\ w \geqslant u$$[8]

Set $v_n := u_n^+ \wedge w$ where $u_n \in B_0$, $u_n \xrightarrow{\|\cdot\|_\infty} u$.

Now $\quad \tilde{p}_t v_n = \tilde{T_t v_n}, \tilde{p}_t v_n \xrightarrow[(DOM)]{a.e.} \tilde{p}_t u$

Mind $\quad v_n \in B_0$, so $\tilde{T_t v_n}$ makes sense.

Is $\tilde{p}_t u$ the q.c. modification of $T_t u$ or $\tilde{p}_t u = \tilde{T_t u}$, reps.?

Idea $\quad T_t v_n \xrightarrow{\tilde{\mathcal{E}}^s_\alpha} T_t u$, then 8.16 $\exists \tilde{T}_t u = \lim_n \tilde{T_t v_n}$ and is q.c. modification of $T_t u$[9]

$$\mathcal{E}^s_{\alpha_0}\left(T_t(v_n - u)\right) = \langle \mathbf{A} T_t(v_n - u), T_t(v_n - u)\rangle_{L^2(m)} + \alpha_0 \langle T_t(v_n - u), T_t(v_n - u)\rangle_{L^2(m)}$$
$$\leqslant C_{t,\gamma} \left\| v_n - u \right\|_{L^2(m)} \xrightarrow[(DOM)]{n\uparrow} 0,$$

and, hence,

$$\begin{array}{ccc} \tilde{p}_t v_n = \tilde{T_t v_n} & \xrightarrow[\text{uniformly 8.16}]{q.e.} & \tilde{T}_t u \text{ q.c. version of } T_t u \\ \Big\downarrow a.e. & & \\ \tilde{p}_t u & & \end{array}$$

Now $\quad \tilde{p}_t u \overset{\text{a.e.}}{=} \tilde{T_t u} \longrightarrow \tilde{p}_t u \overset{\text{q.e}}{=} \tilde{T - tu}.$

7° **Claim** $\quad \forall u \in L^2(m) \cap \mathcal{B}(X) : \tilde{p}_t u = \tilde{T_t u}$

Need $\quad$ MCT = Monotone class theorem

Define $\mathcal{H} = \left\{ u \in L^2(m) \cap \mathcal{B}(X) : \tilde{p}_t u = \tilde{T_t u} \right\}$

6° $\Longrightarrow C_c(X) \subset \mathcal{H} \Longrightarrow$ 10.4 (c) (Urysohn)
$\mathcal{H}$ linear $\Longrightarrow$ 10.4 (a)

---

[9] T_t analytic, i.e. $T_t u \in \mathcal{D}(\mathbf{A})$, $\left\|\mathbf{A} T_t\right\|^{L^2 \to L^2} \leqslant \frac{c}{t}$.

For 10.4 (b): $(u_n)_n \subset \mathcal{H}$, $u_n \uparrow u \in L^2(m) \cap \mathcal{B}(X)$

$$\Longrightarrow u_n \xrightarrow[L^2]{n\uparrow\infty} u, \text{ (DOM)}$$
$$T_t u_n \xrightarrow[n\uparrow\infty]{\mathcal{E}_{\alpha_0}} T_t u, \text{ analytic semigroup, cf. } 6^{\circ}$$

8.16 $\Longrightarrow$

$$\begin{array}{ccc}
\widetilde{T_t u_{n(k)}} & \xrightarrow[\text{uniformly}]{\text{q.e.}} & \widetilde{T_t u} \\
\Big\| \text{ q.e.} & & \Big\| \text{ a.e.} \overset{8.10\,(b)}{\Longrightarrow} \\
\tilde{p}_t u_{n(k)} & \xrightarrow[\text{BL}]{\text{q.e.}} & \tilde{p}_t u
\end{array}$$

$\Longrightarrow u \in \mathcal{H}$
MCT $\Longrightarrow \mathcal{H} = L^2(m) \cap \mathcal{B}(X)$.

■

Remark R_α, $\alpha > \gamma$ goes in the same way.

hp-104 **10.4 Lemma** (MCT). $\mathcal{H} \subset L^2(m) \cap \mathcal{B}(X)$

(a) $\mathcal{H}$ *linear space*

(b) $u_n \in \mathcal{H}, u_n \uparrow u \in L^2(m) \cap \mathcal{B}(X) \Longrightarrow u \in \mathcal{H}$

(c) $\forall U \subset X$ *open* $\exists (u_n)_n \subset \mathcal{H}, u_n \uparrow \mathbb{1}_U$ *pointwise* $\Longrightarrow \mathcal{H} = L^2(m) \cap \mathcal{B}(X)$.

Proof. $G \subset X$ open, $m(G) < \infty$. Set $\mathcal{D}_G := \{A \subset G : A \text{ Borel}, \mathbb{1}_A \in \mathcal{H}\} \overset{(c)}{\neq}$

Note $\mathcal{O} \cap G = \{U \cap G : U \text{ open}\} \subset \mathcal{D}_G$

(a) - (c) yields: $\mathcal{D}_G$ is a Dynkin-system

$\Longrightarrow \sigma(\mathcal{O} \cap G) = \delta(\mathcal{O} \cap G) \subset \delta(\mathcal{D}_G) = \mathcal{D}_G \subset \sigma(\mathcal{O} \cap G)$
$\Longrightarrow \mathcal{D}_G = \mathcal{B}(G)$

If $u \in L^2_+(m) \cap \mathcal{B}(X)$. Pick open sets $G_n \uparrow X$, $m(G_n) < \infty$. Then by the Sombrero-Lemma: $u\mathbb{1}_{G_n} \in \mathcal{H}$ and $u\mathbb{1}_{G_n} \uparrow u \overset{(b)}{\Longrightarrow} u \in \mathcal{H}$. So $L^2_+(m) \cap \mathcal{B}(X) \subset \mathcal{H} \Longrightarrow L^2(m) \cap \mathcal{B}(X) \subset \mathcal{H}$. ■

hp-105 **10.5 Lemma** (Urysohn). *$K \subset U \subset X$, K compact, U open, $\overline{U}$ compact. Then*

$$\exists f_{K,U} \in C_c(X) : f_{K,U}\big|_K = 1,\ f_{K,U}\big|_{U^c} = 0.$$

Moreover, $\forall K$ compact $\exists U_n$ open, $\overline{U_n}$ compact and $U_n \downarrow K$.

Sketch of the Proof. Set $f_{K,U}(x) = \dfrac{d(x,U^c)}{d(x,U^c) + d(x,K)}$, $d(x,A) = \int_{a \in A} d(x,a)$ is Lipschitz.

To get $U_n \downarrow K$[10] (in X any relatively compact set). Use open cover, take finite sub-cover. ■

(II) Construct a joint nest $\{F_k\}_k$

$(U_n)_{n\in\mathbb{N}}$ is a **basis** of the topology $\mathcal{O}$ in (X,d)[11], if $\forall U \in \mathcal{O}\ \forall x \in U\ \exists n(x) \in \mathbb{N} : x \in U_{n(x)} \subset U$.

$$\begin{aligned}
\mathcal{O}_1 &:= \left\{U_{n_1} \cup \ldots \cup U_{n_k} : n_1 < n_2 < \ldots < n_k,\ k \in \mathbb{N}\right\} \\
e_u &= \text{ equilibrium potential of } U \in \mathcal{O}_1 \\
\tilde{e}_u &\text{ (a) q.c. modification of } e_U \in \mathcal{F},\ 0 \leqslant \tilde{e}_U \leqslant 1 \text{ q.e.}^{12}
\end{aligned}$$

hp-106 **10.6 Definition.** $\tilde{\mathcal{H}}$ is the smallest family in $\tilde{\mathcal{F}} \cap \mathcal{B}_b(X)$ such that

(H1) $B_0 \cup \{\tilde{e}_U\}_{U\in\mathcal{O}_1} \subset \tilde{\mathcal{H}}$ (B_0 from 10.3, 1°)

(H2) $\tilde{p}_t(\tilde{\mathcal{H}}) \subset \tilde{\mathcal{H}}$, $\tilde{R}_\lambda(\tilde{\mathcal{H}}) \subset \tilde{\mathcal{H}}\ \forall t \in \mathbb{Q}_+$, $\lambda \in \mathbb{Q}_+$, $\lambda > \gamma$

(H3) $\tilde{\mathcal{H}}$ is an algebra over $\mathbb{Q}$.

hp-107 **10.7 Lemma.** *$\tilde{\mathcal{H}}$ is countable.*

Proof. Consider the maps $(\tilde{\mathcal{F}} \cap \mathcal{B}_b(X)) \to \tilde{\mathcal{F}} \cap \mathcal{B}_b(X)$

$$\begin{aligned}
s_1(u,v) &= u + v & s_{4,t}(u,v) &= \tilde{p}_t u \\
s_{2,a}(u,v) &= a \cdot u & s_{5,\lambda}(u,v) &= \tilde{R}_\lambda u \\
s_3(u,v) &= u \cdot v
\end{aligned}$$

[10] In $\mathbb{R}^n$: one would take $K + B_{\frac{1}{n}}(0)$

[11] Mind: You get any open set from this basis

Use 10.2 for

$$\mathbb{B} = B_0 \cup \{\tilde{e}_U\}_{U \in \mathfrak{O}_1}$$
$$\Sigma = \{s_1, s_{2,a}, s_4, s_{4,t}, s_{5,\lambda} : a \in \mathbb{Q},\ \lambda, t \in \mathbb{Q}_+\}$$

■

Assumption $\gamma < 1 \longrightarrow$ can take α_0 or $\alpha = 1$ with reference capacity is cap_1 (instead of $\frac{\text{cap}_\alpha}{\text{cap}_{\alpha_0}}$)

hp-108 **10.8 Lemma.** *$\mathbf{A}$ generator of T_t (or $\mathbf{G}_\lambda$ or $\mathcal{E}$), $u \in \mathcal{D}(\mathbf{A}) \subset \mathcal{F}$, in particular, $u = \mathbf{G}_\lambda f$ ($f \in L^2(m)$, $\lambda > \gamma$), then*

(a) $T_t u \xrightarrow[t\downarrow 0]{\mathcal{E}_1^s} u$

(b) $\frac{1}{t}\left(\mathbf{G}_1 u - e^{-1\cdot t}\mathbf{G}_1 T_t u\right) \xrightarrow[t\downarrow 0]{\mathcal{E}_1^s} u$

Proof. (a) $u \in \mathcal{D}(\mathbf{A}) \underset{\mathcal{D}(\mathbf{A}) = \mathbf{G}_\lambda(L^2),\ \lambda > \gamma}{\overset{3.6}{\Longrightarrow}} \exists f \in L^2(m) : u = \mathbf{G}_1 f$

$$\mathcal{E}_1\left(\mathbf{G}_1 T_t f - \mathbf{G}_1 f\right) \underset{(??)}{\overset{??}{=}} \langle T_t f - f, \mathbf{G}_1(T_t f - f)\rangle_{L^2}$$
$$\underset{\text{Res. ineq.}}{\overset{\text{CSI}}{\leqslant}} \frac{1}{1-\gamma}\left\|T_t f - f\right\|^2_{L^2(m)} \xrightarrow{t\downarrow 0} 0$$

(b) $u \in \mathcal{D}(\mathbf{A})$

$$\frac{1}{t}\left(\mathbf{G}_1 u - e^{-t}\mathbf{G}_1 T_t u\right) - \underbrace{\mathbf{G}_1(1-\mathbf{A})}_{=\text{ id}} u$$
$$= \frac{1}{t}\mathbf{G}_1\left(u - T_t u\right) + \frac{1}{t}\left(1 - e^{-t}\right)\mathbf{G}_1 T_t u - \mathbf{G}_1(1-\mathbf{A})u$$
$$\xrightarrow{t\downarrow 0} \mathbf{G}_1(-\mathbf{A})u + \mathbf{G}_1 u - \mathbf{G}_1(1-\mathbf{A})u = \mathbf{G}_1 0.$$

Rest as in (a).

■

hp-109 **10.9 Lemma.** *$\exists$ a regular nest $\{F_k\}_{k\in\mathbb{N}}$ such that $\forall x \in E_1 := \bigcup_k F_k$, $\forall \lambda, t \in \mathbb{Q}_+$, $u \in \tilde{\mathfrak{H}}, U \in \mathfrak{O}_1$ we have*[13]

(a) $\tilde{\mathfrak{H}} \subset \mathcal{C}_\infty\left(\{F_k\}\right)$, $F_k \subset F_k^0$

(b) $\tilde{e}_U(x) = 1\ \forall u \in U \cap E_1$[14]

[13] Definition of E_1 referes to 8.2 (h).
[14] Note that «quasi» is covered by $X \setminus E_1$.

(c) $\exists (t_k)_k \subset \mathbb{Q}_+$, $t_k \downarrow 0$ *and* $\exists(\lambda_k)_k \subset \mathbb{Q}_+$, $\lambda_k \uparrow 0$ *with*

- $\tilde{p}_{t_k}\tilde{R}_\lambda u(x) \xrightarrow{k\uparrow\infty} \tilde{R}_\lambda u(x)$
- $\frac{1}{t_k}\left(\tilde{R}_1\tilde{R}_\lambda u(x) - e^{-t_k}\tilde{R}_1\tilde{p}_{t_k}\tilde{R}_\lambda u(x)\right) \xrightarrow{k\uparrow\infty} \tilde{R}_\lambda u(x)$
- $\lambda_k\tilde{R}_{\lambda_k}u(x) \xrightarrow{k\uparrow\infty} u(x)$

(d) $\tilde{p}_t\tilde{p}_s u(x) = \tilde{p}_{t+s}u(x)$

(e) $\tilde{p}_t\tilde{R}_\lambda u(x) = \tilde{R}_\lambda\tilde{p}_t u(x)$ *and* $e^{-t}\tilde{p}_t\tilde{R}_\lambda u(x) \leqslant \tilde{R}_\lambda u(x)$ *(*$u \geqslant 0$*)*[15]

(f) $e^{-t}\tilde{p}_t\tilde{e}_U(x) \leqslant \tilde{e}_U(x)$

(g) $0 \leqslant \tilde{e}_U(x) \leqslant 1$

(h) $\tilde{e}_U(x) \leqslant \tilde{e}_W(x)\ \forall U \subset W \in \mathcal{O}_1$

Proof. (b) Know from 8.5 (c) $\forall U \in \mathcal{O}_1$

$$e_U \overset{\text{a.e.}}{=} 1 \text{ on } U \overset{8.10\text{ (b)}}{\Longrightarrow} \tilde{e}_U \overset{\text{q.c.}}{=} 1 \text{ on } U \text{ is cts fn}$$

Now $\sharp\mathcal{O}_1 = \sharp\mathbb{N}$ we get

$$\exists N_1,\ \text{cap}_1(N_1) = 0 \forall U \in \mathcal{O}_1\ \forall x \in U \setminus N_1 \,:\, \tilde{e}_U(x) = 1,$$

where $N_1 = \bigcup_{U\in\mathcal{O}_1}$ (U-dependent exceptional sets).

(c) Want to use 8.16. Observe

$$T_{t_k}\mathbf{G}_\lambda u \underset{10.8}{\overset{\mathcal{E}_1^s}{\longrightarrow}} \mathbf{G}_\lambda u,\ \text{some } t_k \downarrow 0,\ \text{fixed } u, \lambda$$

Taking further subsequence of $\{t_k\}$ with the same name, we get[16]

$$\overset{8.16}{\Longrightarrow} \tilde{p}_{t_k}\tilde{R}_\lambda u \underset{uniformly}{\overset{q.e.}{\longrightarrow}} \tilde{R}_\lambda u$$

Assertion $\sharp 2$ is similar (use 10.8 (b)), assertion $\sharp 3$ also similar, use 3.5 (c), i.e.

$$\lambda\mathbf{G}_\lambda u \underset{\lambda\uparrow\infty}{\overset{\mathcal{E}_1^s}{\longrightarrow}} u \oplus\ \text{8.16 as before.}$$

So far $(t_k), (\lambda_k), u, \lambda$ fixed and the exceptional sets depend on these «parameters.»

[15] Only for the second part.

[16] Note that we also using $\mathbb{G}_\lambda u \overset{\text{a.e.}}{=} \tilde{R}_\lambda u$ and $T_{t_k}\mathbf{G}_\lambda u = T_{t_k}\tilde{R}_\lambda u$ as T_{t_k} is an operator on $L^2(m)$.

(1) $\sharp\tilde{\mathcal{H}}$, $\sharp\mathbb{Q}_+ = \sharp\mathbb{N}$, i.e. can take (by diagonal trick) *one* $(t_k)_k$ and *one* $(\lambda_k)_k$ for all u, all $\lambda \in \mathbb{Q}_+$. Take $N_2 := \bigcup_{u,\lambda}$ (u and λ dependent exceptional sets) = common cap_1-null-set for (c).

Now

(a) Corollary 8.10 tells us: $\exists$ common nest $\{F_k\}_{k\in\mathbb{N}}$ for all $u \in \tilde{\mathcal{H}}$ s.t. $\tilde{\mathcal{H}} \subset \mathcal{C}_\infty(\{F_k\})$.

Now $\quad F_k \longrightarrow F_k \cap F_k^0$ is still a nest (F_k^0 from (I))

Can assume $\bigcup_k F_k =: E_1$ satisfies $X \setminus E_1 \supset N_1 \cup N_2 \Longrightarrow$ our common exceptional set is $X \setminus E_1$.

Clear $\quad$ (b) and (c) hold for all $x \in E_1$

(d), (e) $\tilde{p}_t\tilde{p}_s u \overset{\text{a.e.}}{=} T_tT_s u \overset{\text{a.e.}}{=} T_{t+s}u \overset{\text{a.e.}}{=} \tilde{p}_{t+s}u$, but $\tilde{p}_t\tilde{p}_s u$, $\tilde{p}_{t+s}u$ are cts on E_1, so $\tilde{p}_t\tilde{p}_s u = \tilde{p}_{t+s}u$ on E_1 (q.e. on X). (e) very similar, DIY.

(f) e_U is 1-excessive $\Longleftrightarrow e^{-t}T_te_U - e_U \leqslant 0$ a.e. By 8.10 (b) $e^{-t}\tilde{p}_te_U - \tilde{e}_U \leqslant 0$ a.e. As $e_U \overset{\text{a.e.}}{=} \tilde{e}_U$ and $T_te_U \overset{\text{a.e.}}{=} T_t\tilde{e}_U$ we also have $e^{-t}\tilde{p}_t\tilde{e}_U - \tilde{e}_U \leqslant 0$ a.e.

(g), (h) Similar. ■

Need $\quad E_1$, $\tilde{R}_\lambda$, $\tilde{p}_t$ is a semigroup.

(III) Construction of a Markov transition function

hp-1010 **10.10 Lemma.** *(a)* $\exists$ *Borel* $E_1 \subset E_2$ *with* $\mathrm{cap}_1(X \setminus E_2) = 0$[17] *s.t.* $\forall x \in E_2, t \in \mathbb{Q}_+$

$$\tilde{p}_t(x, X \setminus E_2) = 0.$$

(i.e. 1-step transition function a.s. only sees E_2)

(b)

$$p_t(x,B) := \begin{cases} \tilde{p}_t(x,B) & x \in E_2,\ B \in \mathcal{B}(X) \\ 0 & x \notin E_2,\ B \in \mathcal{B}(X) \end{cases}$$

defines ($t \in \mathbb{Q}_+$) a family of Markov sub-probability kernels satisfying Chapman-Kolmogorov, i.e.

$$p_tp_su(x) = p_{t+s}u(x) \quad \forall s,t \in \mathbb{Q}_+,\ u \in \tilde{\mathcal{H}}\ \forall x \in X \tag{10.2}$$ hp::eq05

(i.e. the kernels love on $(X, \mathcal{B}(X))$).

[17] i.e. I can extend my capacity on an exceptional set

Proof. (a) Now that

$$\operatorname{cap}(X \setminus E_1) = 0 \overset{8.11}{\Longrightarrow} m(X \setminus E_1) = 0 \Longrightarrow \mathbb{1}_{X \setminus E_1} = 0 \text{ a.e.} \Longrightarrow T_t \mathbb{1}_{X \setminus E_1} \overset{T_t \text{ in } L^2}{=} 0 \text{ a.e.}$$

Lemma 10.3 (b) says $\tilde{p}_t \mathbb{1}_{X \setminus E_1}(\cdot)$ as a q.c. modification of $T_t \mathbb{1}_{X \setminus E_1}(\cdot)$, so new cap_1-null set. Hence, $\exists$ Borel $E_1^{(1)} \subset E_1$ and $\operatorname{cap}_1(X \setminus E_1^{(1)}) = 0$ and $\forall x \in E_1^{(1)}, t \in \mathbb{Q}_+$: $\tilde{p}_t(x, X \setminus E_1) = 0$. Now iterate this argument. We get

$$E_1 \supset E_1^{(1)} \supset E_2^{(2)} \supset \dots \supset E_1^{(k)} \dots \text{ Borel}$$

such that

$$\forall x \in E_1^{(k+1)},\ t \in \mathbb{Q}_+ : \operatorname{cap}_1(X \setminus E_1^{(k+1)}) = 0 \wedge \tilde{p}_t(x, X \setminus E_1^{(k)}) = 0.$$

Now

$$E_2 := \bigcap_{k \in \mathbb{N}} E_1^{(k)} \text{ does the job.}$$

(b) $E_2 \subset E_1$, so only 10.2 to show. Recall

$$p_t u(x) = \int_{y \in X} u(y) p_t(x, \mathrm{d}y).$$

Therefore,

$$\begin{aligned} p_t p_s u(x) &\underset{x \in E_2}{\overset{\text{def}}{=}} \tilde{p}_t p_s u(x) \\ &\overset{\text{(a)}}{=} \tilde{p}_t \tilde{p}_s u(x) \\ &\underset{x \in E_2}{\overset{10.9}{=}} \tilde{p}_{t+s} u(x) \overset{\text{def}}{=} p_{t+s} u(x) \text{ on } E_2 \end{aligned}$$

■

Problem $p_t(x, X) \leqslant 1$, not $= 1$

Need $X_\Delta = X \cup \{\Delta\}$ **1-point-compactification**

Want $p_t'(x, X_\Delta) = 1$ by $X_\Delta = X \cup \{\Delta\}$, Δ gobbles up all mass defects.

Now Standard trick to go from sub-probability kernel ($p_t(x, X) \leqslant 1$) to a probability kernel ($p_t'(x, X) = 1$)

$$\begin{cases} p_t'(x, B) := p_t(x, B \setminus \{\Delta\}) + \left[1 - p_t(x, X)\right] \delta_\Delta(B), & x \in X,\ B \in \mathcal{B}(X_\Delta),\ t \in \mathbb{Q}^+ \\ p_t'(\Delta, B) = \delta_\Delta(B) \end{cases}$$

(10.3) hp::eq06

$\Longrightarrow p_t'(x, X_\Delta) = 1 \ \forall x \in X_\Delta$ is still a kernel

(IV) Construction of Markov process (on $\mathbb{Q}^+$)

Stochastic basis

$$\Omega_0 = (X_\Delta)^{\mathbb{Q}^+} = \left\{\omega : \omega : \mathbb{Q}^+ \to X_\Delta\right\}, \ \text{shifts} \exists \vartheta_t \omega = \omega(t + \cdot) \tag{10.4}$$ hp::eq07

$$X_t^0 : \Omega_0 \to X_\Delta, \ \omega \mapsto X_t^0(\omega) \overset{\text{def}}{=} \omega(t) \tag{10.5}$$

$$\mathcal{A} = \sigma(X_t^0 : t \in \mathbb{Q}^+) \tag{10.6}$$

$$\mathcal{A}_t^0 = \sigma(X_s^0 : s \leqslant t, \ s \in \mathbb{Q}^+), \ t \in \mathbb{Q}^+ \tag{10.7}$$

$$\mathcal{A}_t := \bigcup_{r>t} \mathcal{A}_r, \ r \in \mathbb{Q}^+, \ t \in \mathbb{R}^+ \tag{10.8}$$

$$\mathcal{A}_t' := \sigma(\mathcal{A}_t, \mathcal{N}), \ \mathcal{N} \text{ all } \mathbb{P}^x\text{-null sets, all } x, \text{ in } E_2 \tag{10.9}$$

(10.10)

hp-1011 **10.11 Lemma.** *$\exists$ a Markov process (with shifts ϑ_t) $(\Omega_0, \mathcal{A}, \mathbb{P}^x, x \in X_\Delta, \mathcal{A}_t^0, X_t, t \in \mathbb{Q}^+, X_\Delta, \mathcal{B}(X_\Delta))$ satisfying $(M1) - (M4)$ of § 9 relativ to $t \in \mathbb{Q}^+$ with transition function $p_t'(x, \mathrm{d}y)$ as in* (10.3).

Proof. Kolmogorov's theorem 9.2 and 9.1, since 9.1 gives us the fdd $(p_{t_1}, ..., p_{t_n})$ from $p_t'(x, \mathrm{d}y)$. ∎

Aim Actual state space is $(E_2) \cup \{\Delta\}$, *not* X_Δ

hp-1012 **10.12 Remark.** We have

$$\mathbb{P}^x(X_t^0 \in B) = p_t(x, B), \quad t \in \mathbb{Q}^+, \ x \in X, \ B \in \mathcal{B}(X).$$

Take $x \in E_2 \cup \{\Delta\}$, then

$$\mathbb{P}^x(X_t^0 \in E_2 \cup \{\Delta\}) \overset{\text{def}}{=} p_t'(x, E_2 \cup \{\Delta\})$$
$$\overset{\text{def}}{=} \underbrace{\tilde{p}_t(x, E_2)}_{\text{construction} = \tilde{p}_t(x,X)} + \left[1 - \tilde{p}_t(x, X)\right] \underbrace{\delta_\Delta(E_2 \cup \{\Delta\})}_{=1} = 1$$

$\overset{\sharp\mathbb{Q} = \sharp\mathbb{N}}{\Longrightarrow} \mathbb{P}^x(X_t^0 \in E_2 \cup \{\Delta\}, \ \forall t \in \mathbb{Q}^+) = 1, \ x \in E_2 \cup \{\Delta\}$. So box $E_2 \cup \{\Delta\}$ is «invariant» for X_t^0

hitting times

$$\sigma^0_U = \inf\left\{t \in \mathbb{Q}^+ : X^0_t \in U\right\}, \quad U \in \mathcal{O}(X)$$

$$\sigma^0_{U\big|_D} := \min\left\{t \in D : X^0_t \in D\right\}, \quad D \subset \mathbb{Q}^+, \text{ finite,}$$

as usual $\inf \varnothing = \min \varnothing := +\infty$.

hp-1013 **10.13 Lemma.** *$U \in \mathcal{O}_1$ ((II), unions of basis sets) and $\tilde{e}_U$. Then*

$$\mathbb{E}^x e^{-\sigma^0_U} \leqslant \tilde{e}_U(x), \quad \forall x \in E_2$$

Proof. Set $Y^0_t(\omega) := e^{-t}\tilde{e}_U(X^0_t(\omega))$ $\omega \in \Omega_0$, $t \in \mathbb{Q}^+$.

1° $(Y^0_t, \mathcal{A}^0_t)_{t\in\mathbb{Q}^+}$ is a positive bounded ($\leqslant 0$) super martingale **Indeed**

- 10.8 (g) $\overset{\text{q.e.}}{\leqslant} \tilde{e}_U \overset{\text{q.e}}{=} 1 \Longrightarrow 0 \leqslant Y^0_t \leqslant 1$ $\mathbb{P}^x$-a.e. $\forall x \in E_2$
- $s \leqslant t$, $s, t \in \mathbb{Q}^+$ and $x \in E_2$

$$e^{-(t-s)}p_{t-s}\tilde{e}_U(x) \overset{x\in E_2}{=} e^{-(t-s)}\tilde{p}_{t-s}\tilde{e}_U(x) \overset{10.8\text{ (f)}}{\leqslant} \tilde{e}_U(x)$$

Thus,

$$\begin{aligned}
\mathbb{E}^x(Y^0_t \mid \mathcal{A}^0_t) &\overset{\text{def}}{=} \mathbb{E}^x(e^{-t}\tilde{e}_U(X^0_t) \mid \mathcal{A}^0_s) \\
&\underset{\mathbb{Q}^+}{\overset{\text{MP on}}{=}} e^{-t}\mathbb{E}^{X^0_s}\tilde{e}_U(X^0_{t-s}) \ \mathbb{P}^x\text{-a.s.} \\
&= e^{-t}\tilde{p}_{t-s}\tilde{e}_U(X^0_s) \ E_2 \text{ a.s. invariant for } X^0_s \\
&\overset{10.8\text{ (f)}}{\leqslant} e^{-s}\tilde{e}_U(X^0_s) = Y^0_s
\end{aligned}$$

2° **Now**[18]

$$\begin{aligned}
\mathbb{E}^x(e^{-\sigma^0_{U|_D}}, \sigma_{U|_D} < \infty) &= \mathbb{E}^x(e^{-\sigma^0_{U|_D}} \underbrace{\tilde{e}_U(X^0_{\sigma^0_{U|_D}})}_{=\ 1 \text{as } X^0_{\sigma^0_{U|_D}} \in D}) \\
&\overset{\text{def}}{=} \mathbb{E}^x Y^0_{\sigma^0_{U|_D}} \\
&\underset{\text{stopping}}{\overset{\text{opt.}}{\leqslant}} \mathbb{E}^x Y_0 \overset{\text{def}}{=} \mathbb{E}^x u(X^0_0) = \tilde{e}_U.
\end{aligned}$$

[18] Note that $\mathbb{E}^x(Z, B) = \mathbb{E}^x(Z\mathbb{1}_B)$.

3^o $D \uparrow \mathbb{Q}^+$, use (DOM) on lhS of the chain of (in)equalities, since[19] $\sigma^0_{U\big|_D} \to \sigma^0_U$.

■

hp-1014 **10.14 Corollary.** *(a) $U \in \mathcal{O}$ and $\tilde{e}_U$ is q.c. version of e_U, then*

$$\mathbb{E}e^{-\sigma^0_U} \leqslant \tilde{e}_U(x) \text{ q.e.}, \qquad (10.11)$$ hp::eq08

i.e. for all $x \in E_2$.

(b) If $U_n \in \mathcal{O}$, $U_n \downarrow$ and $\text{cap}_1(U_n) \downarrow 0$, then

$$\mathbb{P}^x(\lim_n \sigma^0_{U_n} = \infty) = 1 \text{ q.e.}$$

Proof. (b) follows from (a) and Beppo Levi $\text{cap}_1(U_n) \asymp \mathcal{E}_1(e_{U_n}, e_{U_n}) \downarrow 0$, i.e. $e_{U_n} \downarrow 0$ in $\mathcal{E}_1^s$-sense and Lemma 8.16 $\tilde{e}_{U_{n(k)}} \xrightarrow[\text{q.-uniformly}]{n\uparrow\infty} 0$ for a subsequence $(n(k))_k$. Thus, by (10.11),

$$\mathbb{E}^x e^{-\sigma^0_{U_n}} \xrightarrow{\text{n}} 0$$
$$\Longrightarrow \sigma^0_{U_n} \xrightarrow{\text{n}} 0 \ \mathbb{P}^x\text{-a.s.}$$

(a) Have to show (10.11). $U \in \mathcal{O}$ take $(V_n) \subset \mathcal{O}_1$, $V_n \uparrow U$ and by 10.8 $\tilde{e}_{V_n}(x) \uparrow (x \in E_2)$, $\tilde{e}_U := \lim_n \tilde{e}_{V_n}(x)$ on E_2 (Notation!). With Lemma 10.13 and Exercise above and Beppo Levi, (10.11) follows. Formally! Need meaning of $\tilde{e}_U$. Show $\tilde{e}_U$ is q.c. version of e_U = equilibrium potential of U.

1^o

$$\mathcal{E}_1(e_{V_n}) \overset{8.5\,(a)}{\leqslant} \mathcal{E}_1(e_{V_n}, \tilde{e}_{V_n})$$
$$\overset{\text{def}}{=} \text{cap}_1(V_n)$$
$$\leqslant \text{cap}_1(U) \overset{\text{WLOG}}{<} \infty.$$

$\exists e^* \in \mathcal{F} : e_{V_n} \overset{\mathcal{E}_1^s}{r ightharpoone}^* = e_U$ like in Proof of 8.7 (c).

Need $\quad e^* \in \mathcal{L}_U = \left\{\omega \in \mathcal{F} : \omega\big|_U \geqslant 1\right\}$ since $e_{V_n} \uparrow e^*$

[19] Exercise: $U_n \in \mathcal{O}_1$, $U_n \uparrow U_n$: $\sigma^0_{U_n} \uparrow \sigma^0_U$. Normally this requires continuity of the sample path, but we have $t \in \mathbb{Q}^+$.

Now

$$\limsup_n \mathcal{E}_1(e_{V_n}) \underset{e_U = e^* \in \mathcal{L}_U}{\overset{8.5\,(a)}{\leqslant}} \limsup_n \mathcal{E}_1(e_{V_n}, e_U) \underset{\text{conv.}}{\overset{\text{weak}}{=}} \mathcal{E}_1(e_u) \overset{\text{resonance}}{\leqslant} \liminf_n \mathcal{E}_1(e_{V_n})$$

$$\Longrightarrow \lim_n \mathcal{E}_1(e_{V_n}) = \mathcal{E}_1(e_U)$$

2^o $\mathcal{E}_1^s(e_U - e_{V_n}) \to 0$,

$$\mathcal{E}_1^s(e_U - e_{V_n}) = \mathcal{E}_1^s(e_U) - 2 \underbrace{\mathcal{E}_1^s(e_U, e_{V_n})}_{rightharpoon \mathcal{E}_1^s(e_U)} + \underbrace{\mathcal{E}_1^s(e_{V_n})}_{\overset{1^o}{\longrightarrow} \mathcal{E}^s(e_U)}$$

$$\xrightarrow{n\uparrow\infty} 0.$$

$$\overset{8.16}{\Longrightarrow} \tilde{e}_{V_{n(k)}} \xrightarrow[\text{q.e. uniformly}]{\mathcal{E}_1} \tilde{e}_U = \text{q.c. version of } e_U.$$

■

Want to pass $\mathbb{Q}^+ \to \mathbb{R}^+$. «Fill gaps» in X_t^0. Idea

$$X_{t\pm} = \lim_{\substack{s\to t\pm \\ s\in\mathbb{Q}^+}} X_s^0$$

Problem Existence? $\longrightarrow$ martingale theory. ($s \to t+ \iff s \downarrow t$, $s \to t- \iff s \uparrow t$)

hp-1015 **10.15 Lemma.** *$\exists$ Borel $E_3 \subset E_2$ (from Lemma 10.10) with $\operatorname{cap}_1(X \setminus E_3) = 0$ and for all $x \in E_3$:*

(a) $\mathbb{P}^x(\Omega_1) = 1$ where $\Omega_1 = \Omega_{0,1} \cap \Omega_{0,2}$ and

$$\Omega_{0,1} = \left\{ \omega \in \Omega_0 \;:\; \lim_k \sigma^0_{X\setminus F_k}(\omega) = \infty \right\}$$

$$\Omega_{0,2} = \left\{\omega \in \Omega_0 \;:\; \forall t \geqslant 0 \text{ left/right limit of } (X_s^0)_{s\in\mathbb{Q}^+} \text{ exists } \in E_2 \cup \{\Delta\}\right\}$$

(b) Define the process

$$X_t(\omega) := \begin{cases} \lim_{\mathbb{Q}^+\ni s\downarrow t} X_s^0(\omega) & \omega \in \Omega_1,\ t \geqslant 0 \\ 0 & w \notin \Omega_1 \end{cases}$$

Then, $\mathbb{P}^x(X_t = X_t^0\ \forall t \in \mathbb{Q}^+) = 1$.

(c) $\mathbb{P}^x(X_0 = x) = 1$

(d) $\mathrm{Range}(\omega, t) = \{X_s(\omega) \,:\, s \in [0,t]\}$ *and* $\Omega_2 := \{w \in \Omega_1 \,:\, \mathrm{Range}(\omega,t) \subset X$ *compact if* $X_t(\omega) \in X$
Then $\mathbb{P}^x(\Omega_2) = 1$.

Proof. (a) 1° Use Corollary 10.7 (b) $\exists E_3$ as above such that $\mathbb{P}^x(\Omega_0) = 1$ and $\mathbb{P}^x(\Omega_{01}) = 1\ \forall x \in E_3$.

2° **Claim** $\tilde{R}_1(\tilde{\mathcal{H}}^+) \underset{10.7}{\overset{10.6}{\subset}} \mathcal{C}_\infty(\{F_k\})$ separates points in $E_1 \cup \{\Delta\}$

Indeed assume $x, y \in E_1 \cup \{\Delta\}$ and $\tilde{R}_1 u(x) = \tilde{R}_1 u(y)\ \forall u \in \tilde{\mathcal{H}}^+$. So,

$$\tilde{R}_1\tilde{R}_\lambda u(x) - e^{-t_k}\tilde{R}_1\tilde{p}_{t_k}\tilde{R}_\lambda u(x) = \tilde{R}_1\tilde{R}_\lambda u(y) - e^{-t_k}\tilde{R}_1\tilde{p}_{t_k}\tilde{R}_\lambda u(y),$$

since $\tilde{R}_\lambda, \tilde{p}_t$ preserve $\tilde{\mathcal{H}}^+$. So for $(t_k)_k, t_k \downarrow 0$ as in 10.9 (c) $\tilde{R}_\lambda u(x) = \tilde{R}_\lambda u(y)$, multiply by λ and again by 10.9 $\lambda = \lambda_k \uparrow \infty$,

$$u(x) = u(y) \underset{C_c^+ \cap B_0^+ \text{ dense in } C_c^+}{\overset{B_0^+ \subset \tilde{\mathcal{H}}^+}{\Longrightarrow}} x = y,$$

and separates points.

3° **Claim** $\Omega_{01} \setminus \Omega_{02} \subset \bigcup_{u \in \mathcal{H}^+} \Omega_0^{[u]}$, where

$$\Omega_0^{[u]} = \left\{\omega \in \Omega_0 \,:\, \left(\tilde{R}_1 u(X_s^0(\omega))\right)_{s \in \mathbb{Q}^+} \text{ does not have right or left limits for some } t\right\}$$

Indeed Take $\omega \in \Omega_{01} \setminus \Omega_{02}$. Then $\exists t \geqslant 0$

$$X_{t+}^0 := \lim_{\mathbb{Q} \ni s \downarrow t} X_s^0 \text{ or } X_{t-}^0 = \lim_{\mathbb{Q} \ni s \downarrow t} X_s^0 \text{ does not exists.}$$

WLOG $X_{t+}^0 := \lim_{\mathbb{Q} \ni s \downarrow t} X_s^0$ does not exist. Then

$$\forall w \in \Omega_{01}\ \exists k \in \mathbb{N} \,:\, t < \sigma^0_{X \setminus F_k}(\omega)$$

$$\Longrightarrow \exists (s_i)_i,\ (s_i')_i \subset \mathbb{Q}^+,\ s_i \downarrow t,\ s_i' \downarrow t\ \exists x \neq y,\ x, y \in F_k \cup \{\Delta\} \,:$$

$$\lim_i X_{s_i}^0(\omega) = x \neq y = \lim_i X_{s_i'}(\omega)$$

$\overset{2°}{\Longrightarrow} \exists u \in \tilde{\mathcal{H}}^+ \,:\, \tilde{R}_1 u(x) \neq \tilde{R}_1 u(y)$.
Know $\tilde{R}_1 u\big|_{F_k \cup \{\Delta\}}$ is cts (b/o $\mathcal{C}_\infty$, i.e. 0 at $\{\Delta\}$)
$\Longrightarrow \lim_{s_i \downarrow t} \tilde{R}_1 u(X_{s_i}^0(\omega)) \neq \lim_{s_i' \downarrow t} \tilde{R}_1 u(X_{s_i'}^0(\omega)$
$\Longrightarrow$ claim.

4° **Aim** $\mathbb{P}^x(\Omega_0^{[u]}) = 0 \Longrightarrow \mathbb{P}^x(\Omega_{01} \setminus \Omega_{02}) = 0$
$\Longrightarrow \mathbb{P}^x(\Omega_{02}) = 1 \Longrightarrow \mathbb{P}^x(\Omega_1) = 1$.

Fix $x \in E_2$, $u \in \tilde{\mathcal{H}}^+$, $s < t$, $s, t \in \mathbb{Q}^+$.

$$\begin{aligned}
\mathbb{E}^x\left(e^{-t}\tilde{R}_1 u(X_t^0) \mid A_s^0\right) =: \mathbb{E}^x\left(M_t^{[u]} \mid A_s^0\right) \\
&\underset{10.11}{\overset{\text{MP}}{=}} \mathbb{E}^{X_s^0} e^{-t}\tilde{R}_1 u(X_{t-s}^0) \quad \mathbb{P}^x\text{-a.s.} \\
&\overset{\text{def}}{=} e^{-t} p_{t-s}\tilde{R}_1 u(X_s^0) \\
&\underset{10.12}{\overset{X_s^0 \in E_2 \cup \{\Delta\} \text{ a.s.}}{=}} e^{-t}\tilde{p}_{t-s}\tilde{R}_1 u(X_s^0) \quad \text{(a.s.)} \\
&\underset{e^{-t} = e^{-(t-s)}e^{-s}}{\overset{10.9\ (e)}{\leqslant}} e^{-s}\tilde{R}_1 u(X_s^0) = M_s^{[u]}
\end{aligned}$$

$\Longrightarrow \left(M_t^{[u]}, \mathcal{A}_t^0\right)_{t\in\mathbb{Q}^+}$ positive, bdd super martingale

$\Longrightarrow \forall u \in \mathcal{H} : \mathbb{P}^x(\Omega_0^{[u]}) = 0$ b/o (super)mg convergence theorem.[20]

(b) Take $u \in C_\infty(X_\Delta) := \left\{u : X \to \mathbb{R} \text{ cts and } \lim_{x\to\Delta} u(x) \text{ exists.}\right\}$ and $v \in \tilde{\mathcal{H}}^+$, $\lambda, t \in \mathbb{Q}^+$, $x \in E_2$.

$$\begin{aligned}
\mathbb{E}^x(u(X_t^0)\tilde{R}_\lambda v(X_t)) &\underset{\text{càdlàg}}{\overset{\text{(DOM)}}{=}} \lim_k \mathbb{E}^x(u(X_t^0)\tilde{R}_\lambda v(X_{t+t_k}^0)) \\
&\underset{10.9\ (c)}{\overset{\text{MP } 10.11}{=}} \lim_k \mathbb{E}^x(u(X_t^0)p_{t_k}\tilde{R}_\lambda v(X_t^0)) \quad \text{(condition w.r.t } \mathcal{A}_t^0) \\
&= \mathbb{E}^x(u(X_t^0)\tilde{R}_\lambda v(X_t^0))
\end{aligned}$$

Now again 10.9 (c), use $\lambda = \lambda_k \uparrow \infty$ (after multiplying by λ_k etc)
$\Longrightarrow \mathbb{E}^x u(X_t^0)v(X_t) = \mathbb{E}^x u(X_t^0)v(X_t^0)$. $*$

Aim $\quad \mathbb{P}^x(X_t \neq X_t^0) = 0$, $u \otimes v \curvearrowright \mathbb{1}_{\{(x,y):x=y\}}(\cdot,\cdot)$

Know $\quad *$ true $\forall u, v \in \mathcal{H}^+$. Use Lemma 10.4 and proof of 10.3 step 7° (= monotone class argument) to see

$$\text{* holds with } u = \mathbb{1}_B,\ v = \mathbb{1}_{B'} \quad \forall B, B' \in \mathcal{B}(X_\Delta) \qquad (**)$$

Need * on $\mathcal{B}(X_\Delta \times X_\Delta)$, $B \times B' \curvearrowright$ general mble set. Use Dynkin-system trick to get[21]

$$\mathbb{E}^x\mathbb{1}_\Gamma(X_t^0, X_t) = \mathbb{E}^x\mathbb{1}_\Gamma(X_t^0, X_t^0) \quad \forall\Gamma \in \mathcal{B}(X_\Delta \times X_\Delta)$$

If $\Gamma = \operatorname{diag} X_\Delta \times X_\Delta = \left\{(x,x) : x \in X_\Delta\right\} \in \mathcal{B}(X_\Delta \times X_\Delta)$. Hence, $\mathbb{P}^x(X_t = X_t^0) = 1$ $\forall t \in \mathbb{Q}^+$.

[20] Remark: «Only» need martingale convergence for discrete martingales ($\mathbb{Q}^+$ indexed!). So this is a standard trick to show that a Markov process has (a modification with) càdlàg paths.

[21] DIY $\sigma(\mathcal{B}(X_\Delta) \times \mathcal{B}(X_\Delta)) = \mathcal{B}(X_\Delta \times X_\Delta)$, $\mathcal{D}' = $ sets $\in \mathcal{B}(X_\Delta \times X_\Delta)$.

(c) Take 10.9 (c) and $\forall u \in \mathcal{H}^+$, $x \in E_2$. Then

$$\mathbb{E}^x \tilde{R}_\lambda u(X_0) \underset{\text{càdlàg}}{\overset{\text{MP}}{=}} \lim_k p_{t_k} \tilde{R}_\lambda u(x) \underset{10.9\ (c)}{\overset{x\in E_2}{=}} \tilde{R}_\lambda u(x)$$

Multiply by λ, pick $\lambda = \lambda_k \uparrow \infty$, by 10.9 (c). So,

$$\mathbb{E}^x u(X_0) = u(x) \quad \forall u \in \tilde{\mathcal{H}}^+,$$

pick $u_n \downarrow \mathbb{1}_{\{x\}}(x)$.

(d) $\exists K_n$ cpt, $K_n \uparrow X$. Take $u_n \in B_0$ ($\subset C_c(X) \cap \mathcal{F}$, cf. 10.3 (a)). $0 \leqslant u_n \leqslant 1$, $u_n\big|_{K_n} > 0$. Set $v := \sum 2^{-n} u_n > 0, \in C_\infty^+(X)$. So, by 10.9,

- $\tilde{R}_1 v(x) > 0\ \forall x \in E_1$ (use $\tilde{R}_1 u_n > 0$ on $K_n \cap E_1$).
- $e^{-t}\tilde{p}_t \tilde{R}_1 v(x) \leqslant \tilde{R}_1 v(x)\ \forall x \in E_1$
- $\tilde{R}_1 v \in C(\{F_k\})$

Define[22]

$$\Omega_1 \setminus \Omega_2 = \bigcup_{t\in\mathbb{Q}^+} \left\{ \omega \in \Omega_1 : \tilde{R}_1 v(X_t) > 0 \text{ and } \inf_{0\leqslant s\leqslant t} \tilde{R}_1 v(X_s) = 0 \right\}$$

Moreover, define $M_s^{[v]}(\omega) = e^{-s}\tilde{R}_1 v(X_s^0(\omega))$, $s \in \mathbb{Q}^+, \omega \in \Omega_1$. As in 4° $\left(M_s^{[u]}, \mathcal{A}_s^0\right)_{s\in\mathbb{Q}^+}$ positive, bdd, supermg. Hence by Fatou's lemma, each $t \geqslant 0$,

$$M_t := \begin{cases} \lim_{\mathbb{Q}^+ \ni s \downarrow t} M_s^{[v]} & \text{if lim exists} \\ 0 & \text{else,} \end{cases}$$

is right cts, positive, bdd, super mg.[23]
$\Longrightarrow \Omega_1 \setminus \Omega_2 \subset \left\{\omega \in \Omega_1, \exists t \geqslant 0 : M_t(\omega) > 0 \text{ and } \inf_{0\leqslant s\leqslant t} M_s(\omega) = 0\right\}$.

Claim $\mathbb{P}^x(\{\omega \in \Omega_1 : M_t(\omega) > 0 \text{ and } \inf_{0\leqslant s\leqslant t} M_s(\omega) = 0\}) = 0$[24]

Follows by Proposition 10.16.

■

hp-1016 **10.16 Proposition.** *Let $(M_t, \mathcal{A}_t)_{t\geqslant 0}$ be right-cts, positive super-mg and let*

$$\begin{aligned}\tau &= \inf\{t > 0 : M_t = 0 \text{ or } M_{t-} = 0\} \\ &= \inf\{t > 0 : M_t\} \wedge \inf\{t > 0 : M_{t-} = 0\}\end{aligned}$$

be (indeed!) a stopping time. Then $\mathbb{P}(M_t = 0\ \forall t \underset{\geqslant}{>} \tau) = 1$.

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[22] $\tilde{R}_1 v(X_t) > 0$ corresponds to $X_t \in X$, the second condition is equivalent to Range(ω, t) not being compact.

[23] DIY, (DOM) and if lim exists has probability 1 by MCT.

[24] Rationale of this: any positive super-mg reaching 0 stays zero.

Proof. By martingale convergence theorem[25] $\lim_{t\to\infty} M_t$ exists. Hence $M_\infty := 0 \leqslant \lim_{t\to\infty} M_t$ closes the super-mg, i.e. $(M_t, \mathcal{A}_t), 0 \leqslant t \leqslant \infty$ is a super-mg. Set

$$\tau_n := \inf\left\{t > 0 : M_t \leqslant \frac{1}{n}\right\}.$$

Then, clearly, $\tau_n \uparrow \tau$, $\tau_n \leqslant \tau$ and

$$M_{\tau_n}(\omega) = \begin{cases} \leqslant \frac{1}{n} & \omega \in \left\{\tau_n < \infty\right\} \\ = 0 & \omega \in \left\{\tau_n = \infty\right\}. \end{cases}$$

Use optional stopping

$$\forall \sigma \geqslant \tau_n : \mathbb{E}M_\sigma \leqslant \mathbb{E}M_{\tau_n} \leqslant \frac{1}{n}.$$

Take $\sigma := \tau + q$, any $0 < q \in \mathbb{Q}^+$, then letting $n \uparrow \infty$

$$M_{\tau+q} = 0 \text{ a.s. (as } \mathbb{E}M_{\tau+q} = 0),$$

with same exceptional set for all q, i.e. $\mathbb{P}(M_{\tau+q} = 0 \ \forall q \in \mathbb{Q}^+,\ q > 0) = 1$.
$\underset{\text{cts}}{\overset{\text{right}}{\Longrightarrow}} \mathbb{P}(M_{\tau+t} = 0 \ \forall t \geqslant 0) = 1.$ ■

[25] $\sup_t \mathbb{E}\left|M_t\right| < \infty, 0 \overset{\text{supermg}}{=} \sup_t \mathbb{E}M_t^- < \infty$. So super-mg always have the MCT.

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