Lecture Stochastic Calculus

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Contents

Chapter 1

Martingales in continuous time

In this chapter we extend various results about time-discrete martingales (MG for short) to MG with index set $[0, \infty)$. The results have been proven in my lecture on probability theory (winter semester 2008/2009). I refer with Prob-Theo *n.m* on the results in this lecture accordingly.

As usual, let $(\Omega, \mathscr{A}, \mathbb{P})$ a probability space. We commence with the definition of (sub-, super-)MG.

Definition 1.1. Let $(\mathscr{F}_t)_{t\geqslant 0}$ be a filtration, i.e. for $s \leqslant t$, $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{A}$ are sub-σ-algebras. A family $(X_t)_{t\geqslant 0}$ of real-valued random variables (RV) is called $(\mathscr{F}_t)$-**martingale**, if

(a) $X_t \in \mathscr{F}_t$, i.e. X is $\mathscr{F}_t$-measurable, (**adapted**)

(b) $X_t \in \mathrm{L}^1(\mathbb{P})$,

(c) $\mathbb{E}(X_t \mid \mathscr{F}_s) = X_s \qquad \forall s \leqslant t.$

If in (c) holds $\leqslant X_s \rightsquigarrow$ **super-MG**, and $\geqslant X_s \rightsquigarrow$**sub-MG**.

We write $(X_t, \mathscr{F}_t)$ for a (sub-, super-)MG to indicate the filtration.

Obviously, for every sequence $t_1 < t_2 < t_3 < \dots$,

$$Y_n := X_{t_n}, \qquad \mathscr{G}_n := \mathscr{F}_{t_n} \tag{1.1}$$

are (sub-, super-)MG on a discrete time set. For such MG we already know many results from the lecture Prob-Theo and (1.1) is the key in transferring results from the discrete case to a general time set.

However, therefore we require a certain *regularity property*, we will see in a moment. We illustrate our approach in a key result. We say that a process $(X_t)_{t\geqslant 0}$ is right continuous if

$$\mathbb{P}\big(\omega : t \mapsto X_t(\omega) \text{ is right continuous}\big) = 1.$$

(Analogous for other properties.)

Theorem 1.2. *Let $p > 1$ and $(X_t, \mathscr{F}_t)_t$ a right continuous MG (or positive sub-MG resp.) with $\mathbb{E}\left|X_t\right|^p < \infty$. Then, for $1/p + 1/q = 1$,*

$$\begin{aligned}\mathbb{E}\left[\sup_{r\leqslant t}\left|X_r\right|^p\right] &\leqslant q^p \mathbb{E}\left|X_t\right|^p \\ &\leqslant q^p \sup_{r\leqslant t} \mathbb{E}\left|X_r\right|^p .\end{aligned}$$

Proof. Let $(t_j)_{j\in\mathbb{N}} = \mathbb{Q}\cap[0,t]$ (countable!). For every family $I_N := \{u_1 < u_2 < \dots < u_N\} \cup \{t\}$, where $\{u_j : j = 1,\dots,N\} = \{t_j : j = 1,\dots,N\}$, then

$$\mathbb{E}\left[\sup_{u\in I_N} |X_u|^p\right] \leqslant q^p \mathbb{E}|X_t|^p \qquad \forall N,$$

since $(X_u, \mathscr{F}_u)_{u\in I_N}$ is a MG (positive sub-MG). By Beppo Levi's theorem, we find

$$\mathbb{E}\left[\sup_{u\in\mathbb{Q}\cap[0,t]\cup\{t\}} |X_u|^p\right] \leqslant \sup_N \mathbb{E}\left[\sup_{u\in I_N} |X_u|^p\right] \leqslant q^p \mathbb{E}|X_t|^p.$$

Hence, *if we know (!)*, that

$$\sup_{\mathbb{Q}\cap[0,t]\cup\{t\}} |X_r|^p = \sup_{r\leqslant t} |X_t|^p \qquad \mathbb{P}\text{-a.s.} \tag{1.2}$$

the proof is finished. ■

For example, equation (1.2) holds, if $(X_t)_{t\geqslant 0}$ is right continuous almost surely.

In what follows, we show that under the mild additional assumptions a MG always admits a right continuous oder càdlàg[1] modification, i.e.

$$\exists(\tilde{X}_t)_{t\geqslant 0} : \qquad t\mapsto \tilde{X}_t \text{ càdlàg and} \qquad \forall t : \quad \mathbb{P}(X_t = \tilde{X}_t) = 1. \tag{1.3}$$

Example 1.3. The following processes $\mathsf{X} = (X_t)_{t\geqslant 0}$ are càdlàg martingales with respect to their natural (= generated by the process) filtration $\mathscr{F}_t := \mathscr{F}_t^{\mathsf{X}} := \sigma(X_r : r\leqslant t)$:

(a) a Brownian motion (BM) $(B_t)_{t\geqslant 0}$

(b) $B_t^2 - t$, where $(B_t)_t$ is a BM

(c) $M_t = L_t - \mathbb{E}L_t$, where $(L_t)_{t\geqslant 0}$ is a Lévy process (LP) with $\mathbb{E}|X_t| < \infty$

(d) $M_t^\xi := e^{i\xi L_t}e^{t\psi(\xi}$, where $(L_t)_{t\geqslant 0}$ is an LP with characteristic exponent ψ

(e) $M_t := (N_t - \lambda t)^2 - \lambda t$, where $(N_t)_{t\geqslant 0}$ is a Poisson process (PP) with intensity $\lambda > 0$

(f) $M_t^u := u(L_t) - \int \mathsf{A}u(L_t)\mathrm{d}t$, where $(L_t)_{t\geqslant 0}$ is an LP with generator A and $u\in\mathscr{D}(\mathsf{A})$

Let us show (d) and leave the rest as an exercise. Adaptedness and integrability are clear. Let $s<t$. Then

$$\begin{aligned}
\mathbb{E}\left(M_t^\xi \,\middle|\, \mathscr{F}_s\right) &= \mathbb{E}\left(e^{i\xi L_t + t\psi(\xi)} \mid \mathscr{F}_s\right)\\
&= \mathbb{E}\left(e^{i\xi(L_t-L_s)+(t-s)\psi(\xi)} M_s^\xi \,\middle|\, \mathscr{F}_s\right)\\
&= M_s^\xi\, \mathbb{E}\Big(\underbrace{e^{i\xi(L_t-L_s)+(t-s)\psi(\xi)}}_{\perp\!\!\!\perp \mathscr{F}_s} \mid \mathscr{F}_s\Big)\\
&= M_s^\xi\, \mathbb{E}e^{i\xi(L_t-L_s)+(t-s)\psi(\xi)} = M_s^\xi.
\end{aligned}$$

To construct the càdlàg modification mentioned above, we require stronger assumptions on the underlying filtration.

[1] *french* continue à droite, limité à gauche; right continuous with left limits.

Definition 1.4. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a complete probability space (i.e. $\mathscr{F}$ contains all $\mathscr{F}$-null sets). Then the filtration $(\mathscr{F}_t)_{t\geqslant 0}$ satisfies the **usual conditions**, if

(a) $\mathscr{F}_t = \mathscr{F}_{t+} := \bigcap_{h>0} \mathscr{F}_{t+h}$ **(right continuous)**

(b) $\mathscr{F}_0 \supset \{N \in \mathscr{F} : \mathbb{P}(N) = 0\}$ **(complete)**

Remark 1.5. Showing completeness is, in general, not problematic; showing the right continuity is *much more* subtle. For a LP, we can show (see e.g. [Pro05, Chapter I, Theorem 31, p. 22]) that for $\mathscr{F}_t^X := \sigma(X_s : s \leqslant t)$, the filtration $\mathscr{F}_t := \sigma(\mathscr{F}_t^X, \mathscr{N})$, with $\mathscr{N} = \mathbb{P}$-null sets, is right continuous.

The discrete up-crossing Lemma Prob-Theo 32.2 can be read as follows

Lemma 1.6. *Let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be a sub-MG. Then it holds for any countable set $D = (t_j)_{j\in\mathbb{N}} \subset [0,\infty)$ and for all $a < b$*

$$(b-a)\mathbb{E}U_D(X,[a,b]) \leqslant \sup_{t\in D} \mathbb{E}(X_t - a)^+, \tag{1.4}$$

where

$$U_D(X,[a,b]) = \sup_N \sharp \left\{ \text{upcrossing of } X_{t_1}, \dots, X_{t_n} \text{over } [a,b] \right\}.$$

Proof. For $D_N := \{u_1 < \dots < u_N\} = \{t_j : j = 1, \dots, N\}$, we know (1.4) from Prob-Theo. The assertion now follows by Beppo Levi (LHS) and our Remark 1.5 (RHS), so that

$$\sup_N \sup_{t\in D_N} = \sup_{t\in D}.$$ ■

Theorem 1.7. *Let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be a sub-MG. Then there is $\Omega_0 \subset \Omega$, $\mathbb{P}(\Omega_0) = 1$, such that*

$$\lim_{\mathbb{Q}\ni r\uparrow t} X_r(\omega) \text{ exists} \qquad \forall t > 0, \omega \in \Omega_0, \tag{1.5}$$

$$\lim_{\mathbb{Q}\ni r\downarrow t} X_r(\omega) \text{ exists} \qquad \forall t \geqslant 0, \omega \in \Omega_0. \tag{1.6}$$

Proof. WLOG we show (1.5), (1.6) for $t \in I := [u,v] \subset [0,\infty)$. For all $a < b$, $a, b \in \mathbb{Q}$, we have by (1.4)

$$\begin{aligned}(b-a)\mathbb{E}U_{\mathbb{Q}\cap[u,v]}(X,[a,b]) &\leqslant \sup_{s\leqslant v} \mathbb{E}(X_s - a)^+ \\ &\leqslant \mathbb{E}(X_v - a)^+ \qquad \text{(sub-MG)} \\ &\leqslant \mathbb{E}\left|X_v\right| + |a|,\end{aligned}$$

$$\Longrightarrow \exists \Omega_0 : \quad \forall \omega \in \Omega_0, a < b \in \mathbb{Q} : \quad U_{\mathbb{Q}\cap[u,v]}(X,[a,b]) < \infty.$$

Hence[2], all limits in the theorem exist, cf. proof of the MG convergence theorem Prob-Theo 32.2 (idea: X_t has not ∞ many oscillations). ■

[2]Attention: More precisely, $\Omega_0 = \cap_{a,b\in\mathbb{Q}} \mathbb{Q}_0^{a,b}$.

We now define

$$X_{t+} := \limsup_{\mathbb{Q}\ni r\downarrow t} X_r \qquad \Big[\overset{\text{Theorem 1.7}}{\underset{\text{a.s.}}{=}} \liminf_{\mathbb{Q}\ni r\downarrow t} .\Big]$$

and analogous

$$X_{t-} := \limsup_{\mathbb{Q}\ni r\uparrow t} X_r.$$

Proposition 1.8. *Let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be a sub-MG $\big[$with $(*)\, t \mapsto \mathbb{E}X_t$ right continuous$\big]$. Then it holds $\mathbb{E}\left|X_{t+}\right| < \infty\ \forall t$ and*

$$X_t \leqslant \mathbb{E}\big(X_{t+} \mid \mathscr{F}_{t+}\big) \qquad \big[= \text{ for } (*)\big]. \tag{1.7}$$

Further, $(X_{t+}, \mathscr{F}_{t+})$ is a sub-MG.

Proof. First, show that always $\sup_{r\leqslant t} \mathbb{E}\left|X_r\right| < \infty\ \forall t$, since

$$\begin{aligned} & \left|X_r\right| = X_r^+ - X_r^- = 2X_r^+ - X_r \\ \Longrightarrow \quad & \mathbb{E}\left|X_r\right| = 2\mathbb{E}X_r^+ - \mathbb{E}X_r \\ \Longrightarrow \quad & \mathbb{E}\left|X_r\right| \leqslant 2\mathbb{E}X_r^+ - \mathbb{E}X_0 \qquad \forall r \leqslant t, \end{aligned}$$

because $(X_t^+)_t$ and $(X_t)_t$ are sub-MG. Hence, the claim follows.

In the remainder of the proof, let $t_n \downarrow t$ be decreasing and WLOG $0 \leqslant t \leqslant t_n \leqslant v$ for some $v > 0$.

1° $(X_{t_n}, \mathscr{F}_{t_n})_n$ is a backwards sub-MG and it holds

$$\sup_n \mathbb{E}\left|X_{t_n}\right| < \infty \quad \Longleftrightarrow \quad \lim_{n\to\infty} \mathbb{E}X_{t_n} > -\infty.$$

Indeed: Backwards sub-MG is clear. Moreover,

$$\begin{aligned} \pm\mathbb{E}X_{t_n} \leqslant \mathbb{E}\left|X_{t_n}\right| &= \mathbb{E}X_{t_n}^+ + \mathbb{E}X_{t_n}^+ \\ &= 2\mathbb{E}X_{t_n}^+ - \mathbb{E}X_{t_n} \\ &\leqslant 2\mathbb{E}X_{t_0}^+ - \mathbb{E}X_{t_n} \qquad \text{(sub-MG)}, \end{aligned}$$

and the claim follows.

Remark with $(*)$ it always holds $\mathbb{E}X_{t_n} = \mathbb{E}X_t > -\infty$!

2° with Fatou: $\mathbb{E}\left|X_{t+}\right| \leqslant \liminf_n \mathbb{E}\left|X_{t_n}\right| \leqslant \sup_{r\leqslant v} \mathbb{E}\left|X_r\right| < \infty$

3° $(X_{t_n}, \mathscr{F}_{t_n})$ is ui Since X_{t_n} is a backwards sub-MG and $t_n \downarrow$, also $\mathbb{E}X_{t_n}$ is decreasing and thus $\lim_n \mathbb{E}X_{t_n} > -\infty$ implies the existence of the limit. Hence

$$\forall \varepsilon > 0\, \exists n \geqslant n_0 \qquad \mathbb{E}X_{t_n} \geqslant \mathbb{E}X_{t_{n_0}} - \varepsilon.$$

Then

$$
\begin{aligned}
\int_{|X_{t_n}|>c} |X_{t_n}| \,\mathrm{d}\mathbb{P} &\leqslant \int_{X_{t_n}<-c} \left(-X_{t_n}\right) \mathrm{d}\mathbb{P} + \int_{X_{t_n}>c} X_{t_n} \mathrm{d}\mathbb{P} \\
&\leqslant \int_{X_{t_n}\leqslant -c} X_{t_n} \mathrm{d}\mathbb{P} - \mathbb{E}X_{t_n} + \int_{X_{t_n}>c} X_{t_n} \mathrm{d}\mathbb{P} \\
&\overset{\text{sMG}}{\leqslant} \int_{X_{t_n}\leqslant -c} X_{t_{n_0}} \mathrm{d}\mathbb{P} - \mathbb{E}X_{t_n} + \int_{X_{t_n}>c} X_{t_{n_0}} \mathrm{d}\mathbb{P} \\
&\overset{(*)}{\leqslant} \int_{X_{t_n}\leqslant -c} X_{t_{n_0}} \mathrm{d}\mathbb{P} - \mathbb{E}X_{t_{n_0}} + \varepsilon + \int_{X_{t_n}>c} X_{t_{n_0}} \mathrm{d}\mathbb{P} \\
&\leqslant \int_{|X_{t_n}|\leqslant c} X_{t_{n_0}} \mathrm{d}\mathbb{P} + \varepsilon \\
&= \int_{\{|X_{t_n}|\leqslant c\}\cap\{|X_{t_{n_0}}>R|\}} X_{t_{n_0}} \mathrm{d}\mathbb{P} + \int_{\{|X_{t_n}|\leqslant c\}\cap\{|X_{t_{n_0}}\leqslant R|\}} X_{t_{n_0}} \mathrm{d}\mathbb{P} + \varepsilon \\
&\xrightarrow{R\to\infty} \varepsilon \xrightarrow{\varepsilon\to 0} 0.
\end{aligned}
$$

4° By the backwards sub-MG convergence theorem (PROB-THEO 34.3) it follows $X_{t_n} \to X_{t+}$ a.s. and ui (by Vitali's theorem PROB-THEO 33.14) also in L^1.

5° It holds

$$X_t \overset{\text{sub-MG}}{\leqslant} \mathbb{E}\Big(X_{t_n} \;\Big|\; \mathscr{F}_t\Big) \xrightarrow[\mathrm{L}^1\text{conv.}]{n\to\infty} \mathbb{E}\big(X_{t+} \mid \mathscr{F}_t\big).$$

If $t \mapsto \mathbb{E}X_t$ is right continuous, then also

$$\mathbb{E}X_{t+} = \lim_{n\to\infty} \mathbb{E}X_{t_n} = \mathbb{E}X_t$$

and thus $\mathbb{E}X_t = \mathbb{E}X_{t+}$ and finally $X_t = \mathbb{E}\big[X_{t+} \mid \mathscr{F}_t\big]$ (exercise!).

6° Let $s < s_n < t$ and $s_n \downarrow s$. Then

$$X_{s_n} \overset{\text{sub-MG}}{\leqslant} \mathbb{E}\Big(X_t \mid \mathscr{F}_{s_n}\Big) \overset{5^\circ}{\leqslant} \mathbb{E}\left(X_{t+} \mid \mathscr{F}_t \mid \mathscr{F}_{s_n}\right) \overset{\text{tower}}{=} \mathbb{E}\Big(X_{t+} \mid \mathscr{F}_{s_n}\Big).$$

and both sides converge in L^1 to

$$\text{LHS} \longrightarrow X_{s+} \leqslant \mathbb{E}\big(X_{t+} \mid \mathscr{F}_{t+}\big) \longleftarrow \text{RHS}.$$

(For the RHS: use Lévy convergence theorem PROB-THEO 34.5 for the sub-MG case.) ■

Theorem 1.9. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ satisfy the usual conditions. Let $(X_t)_{t\geqslant 0}$ a sub-MG. If $t \mapsto \mathbb{E}X_t$ is right continuous (e.g. in the MG case!), then $(X_t)_{t\geqslant 0}$ has a càdlàg modification which is again a sub-MG.*

Proof. We set

$$\tilde{X}_t := \begin{cases} \limsup_{\mathbb{Q}\ni r\downarrow t} X_r, & \omega \in \Omega_0 \\ 0, & \omega \notin \Omega_0. \end{cases}$$

with Ω_0 in Theorem 1.7. Obviously, $\tilde{X}_t$ is then right continuous (exercise) and a modification of X_t. Indeed: By (1.7),

$$X_t \overset{(1.7)}{=} \mathbb{E}\left(X_{t+} \mid \mathscr{F}_t\right) \overset{\mathscr{F}_t = \mathscr{F}_{t+}}{=} \mathbb{E}\left(X_{t+} \mid \mathscr{F}_{t+}\right) = X_{t+} = \tilde{X}_t$$

on Ω_0 and $\mathbb{P}\left(\Omega_0^c\right) = 1$ – here we use the completeness. Moreover, by Proposition 1.8, $(\tilde{X}_t)_{t\geqslant 0}$ is a sub-MG and $\tilde{X}_{t-}$ exists by Lemma 1.6, applied to $(\tilde{X}_t, \mathscr{F}_t)$ a.s. ■

Corollary 1.10. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ satisfy the usual conditions. Then, for $Y \in \mathsf{L}^1$, the MG $Y_t := \mathbb{E}\left(Y \mid \mathscr{F}_t\right)$ has a càdlàg modification.*

By the up-crossing lemma, Lemma 1.6, and analogous to Theorem 1.7, we can show the following convergence theorem for sub-MG.

Theorem 1.11. *Let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be a sub-MG. If $\sup_{t\geqslant 0} \mathbb{E}X_t^+ < \infty$, then $\exists \lim_{t\to\infty} X_t$ a.s.*

We also have the following analogous result to Vitali's convergence theorem (Prob-Theo 33.14).

Theorem 1.12. *For a càdlàg MG the following are equivalent:*

(i) $\mathsf{L}^1 - \lim_{t\to\infty} X_t$ *exists;*

(ii) $\exists X_\infty \in \mathsf{L}^1(\mathscr{F}_\infty)$, $\mathscr{F}_\infty := \sigma\left(\bigcup_{t\geqslant 0} \mathscr{F}\right)$, *and* $X_t = \mathbb{E}\left(X_\infty \mid \mathscr{F}_t\right)$*;*

(iii) $(X_t)_{t\geqslant 0}$ *is ui.*

If (i)-(iii) is satisfied, then $X_t \xrightarrow{t\to\infty} X_\infty$ *a.s. and with convergence in* L^1.

Additionally, If $\sup_t \mathbb{E}\left|X_t\right|^p < \infty$ *for some* $p > 0$*, then (iii) holds and* $X_t \xrightarrow{t\to\infty} X_\infty$ *in* L^p.

Proof. (ii)⟹(iii) Prob-Theo Lemma 33.10 b)

(iii)⟹(i) It holds

$$\begin{aligned}
\text{ui} \quad &\Longrightarrow \quad \sup_t \mathbb{E}\left|X_t\right| < \infty \\
&\Longrightarrow \quad \text{Theorem 1.11 holds} \\
&\Longrightarrow \quad X_t \xrightarrow{\text{a.s.}} X_\infty \\
&\Longrightarrow \quad X_{t_n} \xrightarrow[n\to\infty]{\mathsf{L}^1} X_\infty \qquad \forall (t_n),\ t_n \to \infty \qquad \text{(Vitali)} \\
&\Longrightarrow \quad X_t \xrightarrow[t\to\infty]{\mathsf{L}^1} X_\infty.
\end{aligned}$$

(i)⟹(ii) $(X_t)_{t\geqslant 0}$ is MG, i.e.

$$\begin{aligned}
& X_t = \mathbb{E}\left(X_{t+s} \mid \mathscr{F}_t\right) && \forall s > 0 \\
\Longleftrightarrow \quad & \inf_F X_t \mathrm{d}\mathbb{P} = \int_F X_{t+s} \mathrm{d}\mathbb{P} && \forall s > 0,\ F \in \mathscr{F}_t \\
\overset{s\uparrow\infty}{\underset{\mathsf{L}^1}{\Longrightarrow}} \quad & \inf_F X_t \mathrm{d}\mathbb{P} = \int_F X_\infty \mathrm{d}\mathbb{P} && \forall F \in \mathscr{F}_t \\
\Longleftrightarrow \quad & X_t = \mathbb{E}\left(X_\infty \mid \mathscr{F}_t\right).
\end{aligned}$$

Addition

$$\sup_t \mathbb{E}\left|X_t\right|^p < \infty \quad \underset{\text{Doob}}{\Longrightarrow} \quad Y := \sup_{t>0}\left|X_t\right| \in \mathsf{L}^p$$

$$\overset{\text{Prob-Theo}}{\underset{33.11\text{ b)}}{\Longrightarrow}} \quad \left|X_t\right|^p \text{ ui}$$

$$\overset{\text{Vitali}}{\Longrightarrow} \quad X_t \xrightarrow{\mathsf{L}^p} X_\infty. \qquad \blacksquare$$

Finally, we call the discrete Doob decomposition Prob-Theo 30.7. Therefore, let $(X_{t_n}, \mathscr{F}_{t_n})_n$ be a sub-MG. Then there are a MG $(M_n, \mathscr{F}_n)_{n\in\mathbb{N}_0}$ and an increasing predictable (previsible) process $(A_n)_{n\in\mathbb{N}_0}$ (i.e. $A_n \in \mathscr{F}_{n-1}$), with

$$X_n = X_0 + M_n + A_n.$$

This decomposition is unique up to indistinguishability.

We say that two processes X_t, Y_t are **indistinguishable** if

$$\mathbb{P}\big(X_t = Y_t \quad \forall t\big) = 1.$$

In continuous time there is a similar result that (however) is **much harder** to prove. We omit the technical proof, but require this key result later on.

Theorem 1.13. *Let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be a càdlàg-sub-MG of* **class D***, i.e. it holds*

$$\big\{X_\tau \,:\, \tau \text{ is a finite stopping time}\big\} \quad \text{is ui.} \tag{D}$$

Then there is (up to indistinguishability) a unique càdlàg-MG $(M_t)_{t\geqslant 0}$ and a predictable increasing process $(A_t)_{t\geqslant 0}$ such that

$$X_t = X_0 + M_t + A_t \qquad \text{and} \qquad M_0 = A_0 = 0.$$

Definition 1.14. Let $(\mathscr{F}_t)_{t\geqslant 0}$ be filtration. Then we call the random time $\tau : \Omega \to [0,\infty]$ a **stopping time** if $\{\tau \leqslant t\} \in \mathscr{F}_t \quad \forall t \geqslant 0$.

Definition 1.15. The **predictable (previsible) σ-algebra $\mathscr{P}$** is the smallest σ-algebra $\mathscr{P} \subset \mathscr{F} \otimes \mathscr{B}[0,\infty)$ with respect to which all left continuous processes with right limits (càg**l**à**d**) are measurable.

An $\mathscr{P}$-measurable process is called **predictable (previsible)**.

Obviously, left continuity says, e.g. take $\mathbb{1}_{(s,t]}$, that we know the behaviour at time t already «shortly before $t_0 = t$». In other words, an infinitesimal «look into the future».

Chapter 2

Stopping & Localisation

Given a probability filtered space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ (probability space with filtration) satisfying the usual conditions. In this chapter, we study various properties of stopping times (ST).

Proposition 2.1. *Let S, T, T_n be $\mathscr{F}_t$-ST.*

(a) $\{T \leqslant t\} \in \mathscr{F}_t \iff \{T < t\} \in \mathscr{F}_t$,

(b) $S \wedge T$, $S \vee T$ *are ST,*

(c) $\inf_n T_n$, $sup_n T_n$ *are ST,*

(d) For every T there is a sequence of ST $(T_n)_n$ with $T_n \downarrow T$ and every T_n has only countable many values.

Proof. (a) $\implies$ It holds

$$\{T < t\} = \bigcup_k \underbrace{\left\{T \leqslant t - \frac{1}{k}\right\}}_{\in \mathscr{F}_{t-1/k} \subset \mathscr{F}_t} \in \mathscr{F}_t.$$

$\impliedby$

$$\{T < t\} = \bigcap_k \left\{T < t + \frac{1}{k}\right\} \in \bigcap_k \mathscr{F}_{t+\frac{1}{k}} = \mathscr{F}_{t+} = \mathscr{F}_t.$$

(b) follows from (c)

(c) By a direct calculation

$$\left\{\inf_n T_n < t\right\} = \bigcup_n \{T_n < t\} \in \mathscr{F}_t \qquad \text{(wrong for } \leqslant)$$

$$\left\{\sup_n T_n \leqslant t\right\} = \bigcap_n \{T_n \leqslant t\} \in \mathscr{F}_t \qquad \text{(wrong for } <)$$

(d) Set

$$T_n(\omega) := \begin{cases} +\infty, & T(\omega) \geqslant n \\ k2^{-n}, & (k-1)2^{-n} \leqslant T(\omega) \leqslant k2^{-n}. \end{cases}$$

Obviously, $T_n \geqslant T$ and $T_n \downarrow T$; also for $t \in [k2^{-n}, (k+1)2^{-n})$

$$\{T_n \leqslant t\} = \{T_n \leqslant k2^{-n}\} = \{T < k2^{-n}\} \in \mathscr{F}_{k2^{-n}} \subset \mathscr{F}_t. \qquad \blacksquare$$

Definition 2.2. We can associate σ-algebras with an $\mathcal{F}_t$-stopping time τ by

$$\mathcal{F}_T := \left\{A \in \mathcal{F}_\infty : A \cap \{T \leqslant t\} \in \mathcal{F}_t \quad t \geqslant 0\right\}.$$

Note that, despite the slightly misleading notation, F_T is a family of sets and F_T does not depend on ω.

Proposition 2.3. *Let S, T be two $\mathcal{F}_t$-stopping times. Then*

(a) $A \cap \{T < t\} \in \mathcal{F}_t$ *for* $A \in \mathcal{F}_T$, $t \geqslant 0$,

(b) $T \equiv t \quad \Longrightarrow \quad \mathcal{F}_t = \mathcal{F}_T$,

(c) $S \leqslant T \quad \Longrightarrow \quad \mathcal{F}_S \subset \mathcal{F}_T$,

(d) $S \wedge T, T \in \mathcal{F}_T$.

Proof. Exercise. ■

Theorem 2.4. *Let $(X_t)_{t\geqslant 0}$ be an $\mathcal{F}_t$-adapted, càdlàg process. Then*

$$X_T \mathbb{1}_{\{T<\infty\}} \in \mathcal{F}_T.$$

Proof. WLOG, let $\{T < \infty\} = \Omega$. Approximate T by $T_n \downarrow T$ like in Proposition 2.1 d). Then it holds $X_{T_n}\mathbb{1}_{\{T_n<\infty\}} \xrightarrow{n\to\infty} X_T$, as $(X_t)_{t\geqslant 0}$ is càdlàg.

1^{o} $X_{T_n}\mathbb{1}_{\{T_n<\infty\}} \in \mathcal{F}_{T_n}$

$$\begin{aligned} &\Longrightarrow \quad X_T \in \mathcal{F}_{T_n} \qquad \forall n \\ &\Longrightarrow \quad X_T \in \bigcap_n \mathcal{F}_{T_n}. \end{aligned}$$

Indeed: For $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$\{T_n < t\} \cap \left\{X_{T_n} \in B\right\} = \bigcup_k \underbrace{\underbrace{\left\{X_{k2^{-n}} \in B\right\} \cap \left\{T_n = k2^{-n}\right\}}_{\in \mathcal{F}_{k2^{-n}}} \cap \left\{T_n \leqslant t\right\}}_{\subset \mathcal{F}_t} \in \mathcal{F}_t$$

2^{o} $\mathcal{F}_T = \bigcap_n \mathcal{F}_{T_n}$

$\subset$ clear by Proposition 2.3 (c).

$\supset$ Let $A \in \bigcap_n \mathcal{F}_{T_n}$. Then

$$\begin{aligned} A \cap \{T \leqslant t\} &= \bigcap_m A \cap \left\{T < t + \frac{1}{m}\right\} \\ &= \bigcap_m A \cap \left\{\inf_n T_n < t + \frac{1}{m}\right\} \\ &= \bigcap_m \underbrace{\bigcup_n A \cap \left\{T_n < t + \frac{1}{m}\right\}}_{\in \mathcal{F}_{t+1/m} \text{ by assumption}} \in \mathcal{F}_{t+} = \mathcal{F}_t \end{aligned}$$

■

Next, let us show two typical examples for stopping times.

Theorem 2.5. *Let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be an adapted càdlàg process and $U \subset \mathbb{R}^d$ be open. Then* **(first) hitting time of U**

$$T(\omega) := T_U(\omega) := \inf\left\{t > 0 : X_t(\omega) \in U\right\}$$

defines an $\mathscr{F}_t$-ST.

Proof. Using $\mathscr{F}_t = \mathscr{F}_{t+}$, clearly

$$\{T < t\} = \bigcup_{r\in\mathbb{Q}\cap[0,t)} \underbrace{\left\{X_r \in U\right\}}_{\in\mathscr{F}_t} \in \mathscr{F}_t.$$

■

Theorem 2.6. *Let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be an adapted càdlàg process and $F \subset \mathbb{R}^d$ closed. Then*

$$S(\omega) := T_F(\omega) := \inf\left\{t > 0 : X_t(\omega) \in F \vee X_{t-}(\omega) \in F\right\}$$

is an $\mathscr{F}_t$-ST.

Proof. Set $U_n = F + B(0, 1/n)$ («rimmed» set). Then U_n is open and

$$\{S \leqslant t\} = \left\{X_t \in F\right\} \cup \left\{X_{t-} \in F\right\} \cup \bigcap_n \bigcup_{\mathbb{Q}\ni r<t} \left\{X_r \in U_n\right\}.$$

■

More general hitting times (for Borel sets) are also SZ, but this a deep result, by Blumenthal-Getoor, requiring Choquet capacities.

Our next goal is the so called *optional stopping* result by Doob.

Theorem 2.7. *Let $(X_t, \mathscr{F}_t)_{t\geqslant 0} \subset \mathsf{L}^1$ an adapted right continuous process. TFAE:*

(a) $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ *is a MG;*

(b) $\mathbb{E}X_S = \mathbb{E}X_T \qquad \forall$ *a.s. bounded ST* $S \leqslant T$*;*

(c) $X_S = \mathbb{E}\left(X_T \mid \mathscr{F}_S\right) \qquad \forall$ *a.s. bounded ST* $S \leqslant T$.

Note: in (b), (c) we implicitly assume $X_S, X_T \in \mathsf{L}^1$!

Proof. For S, T with only finitely many values, the result is already known from Prob-Theo 31.9.

If S, T are arbitrary and $S \leqslant T \leqslant c < \infty$, then we can find $S_n \downarrow S$, $T_n \downarrow T$ with $S_n \leqslant T_n \leqslant c + 1$ (cf. Proposition 2.1), and $X_{T_n}, S_{T_n} \in \mathsf{L}^1$ (cf. Prob-Theo 31.9).

Since $(X_{S_n}, \mathscr{F}_{S_n})_n$, $(X_{T_n}, \mathscr{F}_{T_n})_n$ are backwards-MG (Prob-Theo 34.3), then

$$X_{S_n} \xrightarrow[\text{a.s.}]{\mathsf{L}^1} X_S, \qquad X_{T_n} \xrightarrow[\text{a.s.}]{\mathsf{L}^1} X_T$$

and (b), (c) follow by taking the limit. ■

We will need some sharper results following from Theorem 2.7.

Corollary 2.8. *Let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be a right continuous ui-MG and $S \leqslant T < \infty$ be finite ST. Then*

$$\mathbb{E}X_S = \mathbb{E}X_T \qquad \text{and} \qquad X_S = \mathbb{E}\big(X_T \mid \mathscr{F}_S\big)$$

Proof. The step $\mathbb{E}X_S \leqslant \mathbb{E}X_T$ for all $S \leqslant T \implies X_S \leqslant \mathbb{E}\big(X_T \mid \mathscr{F}_S\big)$ follows as in PROB-THEO 31.9: Consider $F \in \mathscr{F}_S \subset \mathscr{F}_T$

$$\rho := S\mathbb{1}_F + T\mathbb{1}_{F^c} \leqslant T\mathbb{1}_F + T\mathbb{1}_{F^c} = T.$$

Because ρ is a ST (exercise!), we get

$$\mathbb{E}X_\rho = \mathbb{E}X_T \qquad \implies \qquad \int_F X_S \mathrm{d}\mathbb{P} = \int X_T \mathrm{d}\mathbb{P}.$$

Thus, we only have to show that $X_S, X_T \in \mathsf{L}^1$ and $\mathbb{E}X_S = \mathbb{E}X_T$.

$\boxed{1^o}$ $\exists X_\infty = \mathsf{L}^1 - \lim_{t\to\infty} X_t$ since its ui (Theorem 1.12).

$\boxed{2^o}$ Approximate S, T by S_n, T_n as in Proposition 2.1. Consider

$$T_n = \infty \qquad \iff \qquad T \geqslant n.$$

$\boxed{3^o}$ Let $m > n+1$. If $T_n < \infty$, then $T < n$ and $T_n \leqslant n+1 \leqslant m$. Thus:

$$\begin{aligned}\mathbb{E}\left|X_{T_n\wedge m}\right| &= \mathbb{E}\left[X_{T_n\wedge m}\mathbb{1}_{\{T_n<\infty\}}\right] + \mathbb{E}\left[X_{T_n\wedge m}\mathbb{1}_{\{T_n=\infty\}}\right] \\ &= \underbrace{\mathbb{E}\left[X_{T_n}\mathbb{1}_{\{T_n<\infty\}}\right]}_{\substack{\leqslant \mathbb{E}\sup_{j:T_n=t_j}\left|X_{t_j}\right| \\ \leqslant \sum_{j:T_n=t_j}\mathbb{E}\left|X_{t_j}\right|}} + \underbrace{\mathbb{E}\left[X_{m\wedge m}\mathbb{1}_{\{T_n=\infty\}}\right]}_{\xrightarrow[\text{ui}]{m\to\infty}\mathbb{E}\left[X_\infty\mathbb{1}_{\{T_n=\infty\}}\right]}.\end{aligned}$$

It follows

$$\lim_n \mathbb{E}X_{T_n\wedge m} = \mathbb{E}X_{T_n} \qquad \text{and} \qquad X_{T_n} \in \mathsf{L}^1.$$

$\boxed{4^o}$ $\mathbb{E}X_{T_n\wedge m} = \mathbb{E}X_{S_n\wedge m} \qquad \forall n, m$ by Theorem 2.7.

$$\begin{aligned}&\overset{3^o}{\implies} \qquad \mathbb{E}X_{T_n} = \mathbb{E}X_{S_n} \\ &\overset{m\to\infty}{\implies} \qquad \mathbb{E}X_T = \mathbb{E}X_S.\end{aligned}$$

Indeed: $X_{T_n}, X_{S_n} \in \mathsf{L}^1$, so $X_{T_n} \xrightarrow{\mathsf{L}^1} X_T$, $X_{S_n} \xrightarrow{\mathsf{L}^1} X_S$, since $(X_{T_n})_n$ is a backwards-MG and $(X_t)_{t\geqslant 0}$ is right continuous. ∎

Definition 2.9. Let T be a ST. Then we set

$$X_t^T := X_{t\wedge T} = X_t\mathbb{1}_{\{T>t\}} + X_t\mathbb{1}_{\{T\leqslant t\}}.$$

Corollary 2.10. *Let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be a ui càdlàg MG and T a ST. Then $(X_{t\wedge T}, \mathscr{F}_t)_{t\geqslant 0}$ is again a ui càdlàg MG.*

Proof. Obviously, $t \mapsto X_{t\wedge T}$ is càdlàg. Let $\sigma \leqslant \tau$ be **bounded** $\mathscr{F}_t$-ST. Then also $\sigma\wedge T \leqslant \tau\wedge T$ are bounded ST and, by Theorem 2.7, it follows

$$\mathbb{E}X_{\sigma\wedge T} = \mathbb{E}X_{\tau\wedge T} \qquad \text{or} \qquad \mathbb{E}X_\sigma^T = \mathbb{E}X_\tau^T,$$

i.e., by Theorem 2.7 again, we have that $(X_t^T, \mathscr{F}_t)_{t\geqslant 0}$ is a MG.

Finally, by Corollary 2.8, we have

$$X_{t\wedge T} = \mathbb{E}\big(X_T \bigm| \mathscr{F}_{t\wedge T}\big).$$

Hence X_T closes the MG $(X_{t\wedge T})_{t\geqslant 0}$ and, by Prob-Theo 33.10,

$$\{\mathbb{E}(X \mid \mathscr{C}) : \mathscr{C} \in \mathscr{F}\} \qquad \text{is ui.}$$

■

Example 2.11. The «ui» assumption in Corollary 2.10 is necessary: Consider e.g.

$$M_t := \exp(B_t - t/2) \qquad \big[B_t \quad \text{is a Brownian motion}\big].$$

Then M_t is a MG, $M_0 = 1$. For $\alpha < 1$, let

$$T = T_\alpha = \inf\big\{t : M_t \leqslant \alpha\big\}.$$

It follows: $M_T = \alpha \quad \Longrightarrow \quad \mathbb{E}M_T = \alpha < 1 = \mathbb{E}M_0$.

Attention: $(M_t)_{t\geqslant 0}$ is not ui!

Finally, we show an important result that generalises the notion of a MG.

Definition 2.12. An adapted càdlàg process $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ is a **local martingale** if there is an increasing sequence of ST

$$T_1 \leqslant T_2 \leqslant \ldots \leqslant T_n \leqslant \ldots$$

with $\lim_n \mathbb{P}\big(T_n < t\big) = 0$ for alle $t > 0$, such that

$$(X_t^{T_n}, \mathscr{F}_t)_{t\geqslant 0} \qquad \forall n \qquad \text{is a MG.}$$

The ST $(T_n)_n$ is called **reducing sequence for** X.

Remark 2.13. (a) Every MG is a local MG. Choose $T_n = n$.

(b) Let $(X_t)_{t\geqslant 0}$ be a local MG. Then for every reducing sequence $(T_n)_n$, also $T_n \wedge n$ is a reducing sequence, and $(X_t^{T_n\wedge n})_{t\geqslant 0}$ is an ui MG.
Indeed: The MG property follows from $X_t^{T_n\wedge n} = (X_{t\wedge T_n})^n$ and optimal stopping. The ui property from the relation

$$X_{t\wedge n\wedge T_n} = \mathbb{E}\big(\underbrace{X_{n\wedge T_n}}_{\in \mathrm{L}^1} \mid \mathscr{F}_{t\wedge n\wedge T_n}\big)$$

and Prob-Theo 33.10.

Next, we give a first criterion, when a local MG is a (proper) MG.

Theorem 2.14. *Let* $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ *be a local MG such that* $\mathbb{E}\sup_{r\leqslant t}|X_r| < \infty$ *for all* $t > 0$*. Then* $(X_t)_{t\geqslant 0}$ *is a MG.*
Moreover, if $\mathbb{E}\sup_{r<\infty}|X_r| < \infty$*, then* $(X_t)_{t\geqslant 0}$ *is a ui MG.*

Proof. Let $(T_n)_n$ be a reducing sequence of ST. For $s \leqslant t$, we get

$$\mathbb{E}\Big(X_{t\wedge T_n} \;\Big|\; \mathscr{F}_s\Big) = X_{s\wedge T_n}$$

and with dominated convergence – we have $\left|X_{t\wedge T_n}\right| \leqslant \sup_{r\leqslant t}|X_r| \in \mathrm{L}^1$ – and since $T_n \xrightarrow{\mathbb{P}} \infty$, maybe taking an a.s. convergent subsequence, then

$$\mathbb{E}\big(X_t \mid \mathscr{F}_s\big) = X_s.$$

As $|X_t| \leqslant \sup_{r<\infty}|X_r| \in \mathrm{L}^1$, then by PROB-THEO 33.10, $(X_t)_{t\geqslant 0}$ is ui. ■

Chapter 3

The Stochastic Integral: L^2 theory

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ filtered probability space satisfying the usual conditions. Further, let all (sub-, super-)MG be càdlàg even if not always mentioned explicitly.

An L^2-MG is a MG, which is L^2-**bounded**, i.e.

$$\sup_{t<\infty} \mathbb{E}\left|X_t\right|^2 < \infty. \tag{3.1}$$

We write $\mathscr{M}^2$ for the space of all L^2-MG and $\mathscr{M}_0^2$ for the space of all L^2-MG with $X_0 = 0$.

Doob's maximal inequality, Theorem 1.2, says that

$$\mathbb{E}\sup_{t<\infty}\left|X_t\right|^2 \leqslant 4\sup_{t<\infty}\mathbb{E}\left|X_t\right|^2, \tag{3.2}$$

i.e. $\sup_{t<\infty}\left|X_t\right|^2 =: Y \in \mathrm{L}^1$. Consequently,

$$\left|X_T\right|^2 \leqslant Y \in \mathrm{L}^1 \qquad \forall T \text{ ST},$$

thus, by Prob-Theo 33.11, the family

$$\left\{X_T^2 : T \text{ is ST}\right\} \qquad \text{is ui.}$$

In other words, the condition (D) in the Doob-Meyer decomposition 1.13 is satisfied.

Because $(X_t^2, \mathscr{F}_t)_{t\geqslant 0}$ is a sub-MG, hence

$$X_T^2 = X_0 + M_t + A_t, \tag{3.3}$$

where M = MG and A $\in \mathscr{P}$ and increasing.

Definition 3.1. Let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be a càdlàg L^2-MG. Then we call the uniquely determined, increasing process A with $A_0 = 0$ in the Doob-Meyer decomposition of X^2, (3.3) above, the **(predictable) quadratic variation** or **angle bracket of** X.

We denote it by $\langle X\rangle_t$.

Remark 3.2. (a) $\left(X_t^2 - \langle X\rangle_t\right)_{t\geqslant 0}$ is a MG (with respect to $(\mathscr{F}_t)_{t\geqslant 0}$).

(b) $\mathbb{E}X_t^2 = \mathbb{E}\langle X\rangle_t + \mathbb{E}X_0^2$ (by (a)) and thus $\mathbb{E}\langle X\rangle_t \leqslant \sup_t \mathbb{E}X_t^2$.

(c) For $s \leqslant t$, we have

$$\begin{aligned}
\mathbb{E}\left((X_t - X_s)^2 \mid \mathscr{F}_s\right) &= \mathbb{E}\left(X_t^2 - 2X_tX_s + X_s^2 \mid \mathscr{F}_s\right) \\
&= \mathbb{E}\left(X_t^2 \mid \mathscr{F}_s\right) - 2X_s \underbrace{\mathbb{E}\left(X_t \mid \mathscr{F}_s\right)}_{=X_s} + X_s^2 \\
&= \mathbb{E}\left(X_t^2 - X_s^2 \mid \mathscr{F}_s\right) &(3.4)\\
&= \underbrace{\mathbb{E}\left(X_t^2 - \langle X\rangle_t - (X_s^2 - \langle X\rangle_s) \mid \mathscr{F}_s\right)}_{=0 \text{ by (a), MG}} + \mathbb{E}\left(\langle X\rangle_t - \langle X\rangle_s \mid \mathscr{F}_s\right) \\
&= \mathbb{E}\left(\langle X\rangle_t - \langle X\rangle_s \mid \mathscr{F}_s\right). &(3.5)
\end{aligned}$$

From now on, let $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ be a fixed MG.

Lemma 3.3. *Let* A *be an increasing* $\mathscr{P}$*-measurable process with* $A_0 = 0$ *and for all* $s \leqslant t$

$$\mathbb{E}\big((X_t - X_s)^2 \mid \mathscr{F}_s\big) = \mathbb{E}\big(A_t - A_s \mid \mathscr{F}_s\big).$$

Then: Y $= \langle X \rangle$.

Proof. From our calculation in Remark 3.2 (c), we see that

$$X_t^2 - A_t \qquad \text{is a MG}$$

and the claims follows by uniqueness of the Doob-Meyer decomposition. ■

Next, we want to introduce the stochastic integral with respect to an L^2-MG. As in the case of the Lebesgue integral, we start defining the integral first for *simple integrands*.

Definition 3.4. A process of the form

$$f(s,\omega) := \sum_{j \text{ finite}} \varphi_j(\omega)\mathbb{1}_{(s_j,s_{j+1}]}(s), \tag{3.6}$$

where $0 = s_0 < s_1 < s_2 < \ldots$ with φ_j bounded and $\mathscr{F}_{s_j}$-measurable, is called **simple predictable process (SPP)**.
We denote the family of SPP by $\mathscr{E}$.

Mind an SPP is left continuous, this $\mathscr{P}$-measurable.

Mind $f(s_j,\omega) = f_{j-1}$ for $j = 1,2,3,\ldots$ and $f(0,\omega) = f(S_0,\omega)0$.

On $\mathscr{E}$, we introduce the following L^2-norm

$$\|f\|_{L^2(\langle X \rangle)} := \sqrt{\mathbb{E}\left[\int_0^\infty |f(s)|^2 \,\mathrm{d}\langle X \rangle_s\right]},$$

where X $\in \mathscr{M}^2$ is a fixed L^2-MG.

Remark 3.5. (a) $\mathrm{d}\langle X \rangle_t$ is the Lebesgue-Stieltjes notation for the measure $\mu(\mathrm{d}s)$ defined by

$$\mu(s,t] = \langle X \rangle_t - \langle X \rangle_s, \qquad s \leqslant t.$$

(b) $L^2(\langle X \rangle) = L^2(\mathbb{P} \otimes \mu)$ with μ from (a).

(c) We can interpret the integral as Riemann-Stieltjes sum and

$$\int_0^\infty |f(s)|^2 \,\mathrm{d}\langle X \rangle_s = \sum_j |f(s_j)|^2 \left(\langle X \rangle_{s_{j+1}} - \langle X \rangle_{s_j}\right).$$

One should take Natanson's book to hand and read Chapter VIII.6 about the (Riemann-)Stieltjes integral:

$$\int f \mathrm{d}g := \lim_{|\Pi|\to 0} \sum_{s_j \in \Pi} f(s_j)\left(g(s_{j+1}) - g(s_j)\right),$$

where the main problem is to show the existence of the limits. Note the relation to the Riemann integral, where $g(s) \equiv s$*. Cf. Theorem 3.16 below.*

(d) Example: $\langle B\rangle_t = t$ for a BM B.

(e) Example: $\langle N - \lambda t\rangle_t = \lambda t$ for a PP N with intensity λ.

Definition 3.6. Let $f \in \mathscr{E}$ with representation (3.6). Then

$$f \bullet X_t := I_X(f)_t := \sum_{j<\infty} \varphi_j \left(X_{s_{j+1}\wedge t} - X_{s_j \wedge t}\right), \tag{3.7}$$

$f \bullet X_0 := 0$ is the **stochastic integral of f with respect to the** L^2**-MG** X.
We write $f \bullet X_t$ or $\int_0^t f(s)\,\mathrm{d}X_s$.

Convention 1: $\displaystyle\int_0^t := \int_{0+}^t = \int_{(0,t]}$

Convention 2: $f \bullet X_t^2 := (f \bullet X)_t^2$, i.e. «bullet before the rest», $\bullet$ in $f \bullet X$ **always before** sub- & superscript

Remark 3.7. Since $\varphi_j \in \mathscr{F}_{s_j}$, (3.7) is just the MG transformation, cf. Prob-Theo 30.9 ff.

The following properties are pivotal.

Proposition 3.8. *Let $f \in \mathscr{E}$ and $X \in \mathscr{M}^2$.*

(a) $f \bullet X \in \mathscr{M}_0^2$ *and it holds*

$$\left\langle \int_0^\bullet f(s)\,\mathrm{d}X_s \right\rangle_t = \langle f \bullet X \rangle_t = \int_0^\infty f^2(s)\,\mathrm{d}\langle X\rangle_s \tag{3.8}$$

$$\mathbb{E}\left|\int_0^t f(s)\,\mathrm{d}X_s\right|^2 = \mathbb{E}\left|f \bullet X_t\right|^2 = \mathbb{E}\int_0^\infty f^2(s)\,\mathrm{d}\langle X\rangle_s \leqslant \|f\|^2_{L^2(\langle X\rangle)} \tag{3.9}$$

In addition: For $t = \infty$, then «=» in (3.9).

(b) $f\bullet : \mathscr{M}^2 \to \mathscr{M}^2$ *is linear.*

(c) $\bullet X : \mathscr{E} \to \mathscr{M}^2$ *is linear.*

(d) $\mathbb{E}\left(\left(f \bullet X_t - f \bullet X_s\right)^2 \,\middle|\, \mathscr{F}_s \right) = \mathbb{E}\left(\int_\infty^s f^2(u)\,\mathrm{d}\langle X\rangle_u \,\middle|\, \mathscr{F}_s \right)$

(e) $\|f\|_{L^2(\langle X\rangle)} = \sup_{t<\infty} \left\|f \bullet X_t\right\|_{L^p} \leqslant \|f\|_\infty \cdot \sup_{t<\infty}\left\|X_t\right\|_{L^p}$

(f) $t \mapsto f \bullet X$ *is càdlàg and* $\Delta f \bullet X_t = f(t)\Delta X_t$.[a]

(g) $t \mapsto X_t$ *continuous* $\implies$ $t \mapsto f \bullet X_t$ *is continuous.*

(h) Let τ be a discrete ST. Then

$$(f \bullet X)^\tau = f \bullet (X^\tau) = (f\mathbb{1}_{(0,\tau]}) \bullet X.$$

[a] For a càdlàg process is $\Delta Y_t := Y_t - Y_{t-} = Y_t - \lim_{s\uparrow t} Y_s$.

Proof. (a) We have already seen this result in Prob-Theo 30.11! But the ideas are crucial, we give a detailed proof here.

1° $f \bullet X$ is a MG. Clearly, $f \bullet X$ is in L^1 (even in L^2) and adapted. For $s < t$, we can assume (WLOG), that $s = s_n$ and $t = s_N$ partition points in the representation (3.6). If not, we could add those points. Thus

$$\begin{aligned}\mathbb{E}\left(f \bullet X_t - f \bullet X_s \,\middle|\, \mathscr{F}_s\right) &= \mathbb{E}\left[\sum_n^{N-1} \varphi_j (X_{s_{j+1}} - X_{s_j})\right] \\ &= \sum_n^{N-1} \mathbb{E}\left(\mathbb{E}\left(\varphi_j (X_{s_{j+1}} - X_{s_j}) \,\middle|\, \mathscr{F}_{s_j}\right) \,\middle|\, \mathscr{F}_s\right) = 0,\end{aligned}$$

since

$$\begin{aligned}\mathbb{E}\left(\varphi_j (X_{s_{j+1}} - X_{s_j}) \,\middle|\, \mathscr{F}_{s_j}\right) &\overset{\text{tower}}{=} \varphi_j \mathbb{E}\left(X_{s_{j+1}} - X_{s_j} \,\middle|\, \mathscr{F}_{s_j}\right) \\ &= \varphi_j\left(\mathbb{E}\left(X_{s_{j+1}} \,\middle|\, \mathscr{F}_{s_j}\right) - X_{s_j}\right) = 0\end{aligned}$$

2° L^2-boundedness ... follows immediately from the proof of (3.9).

3° Let $s = s_n < s_N = t$. Then

$$\begin{aligned}\mathbb{E}\big(f \bullet X_t^2 \mid \mathscr{F}_s\big) - f \bullet X_s^2 &\overset{(3.4)}{=} \mathbb{E}\Big(\big(f \bullet X_t - f \bullet X_s\big)^2 \,\Big|\, \mathscr{F}_s\Big) \\ &= \sum_{n}^{N-1} \mathbb{E}\bigg(\Big(f \bullet X_{s_{j+1}} - f \bullet X_{s_j}\Big)^2 \,\bigg|\, \mathscr{F}_s\bigg)\end{aligned}$$

(b) Trivial.

(c) Trivial.

(d) Follows from step **3°** above.

(e) Follows from step **3°** above. Mind that by Beppo Levi

$$\begin{aligned}\|f\|^2_{L^2(\langle X\rangle)} &= \mathbb{E}\int_0^\infty f^2(u)\,\mathrm{d}\langle X\rangle_u \\ &\overset{\text{BL}}{=} \sup_{t>0}\mathbb{E}\int_0^t f^2(u)\,\mathrm{d}\langle X\rangle_u \\ &\overset{(3.9)}{=} \sup_{t>0}\mathbb{E}\big[f \bullet X_t^2\big].\end{aligned}$$

(f) That $f \bullet X$ is càdlàg stems immediately from the Definition 3.4, because all terms are càdlàg. Let now t be fixed, e.g. $t \in (s_j, s_{j+1}]$. Choose $s < t$ so small that $s \in (s_j, s_{j+1}]$. Then

$$f \bullet X_t - f \bullet X_s = \varphi_j(X_t - X_s) = f(s)(X_t - X_s) \xrightarrow{s\uparrow t} f(t-)\Delta X_t,$$

and, since $f(t-) = f(t)$, the claim follows.

(g) Follows from (f) and the fact that $\Delta X_t = 0$.

(h) Let $f(s,\omega) = \sum_j \varphi_j(\omega)\mathbb{1}_{(t_j,t_{j+1}]}(s)$ from $\mathscr{E}$ and τ be a ST with $\tau(\omega) \in \{0, t_1, t_2, \dots\}$ (if not we add additional partition points to f!). Obvisouly,

$$\begin{aligned}(f \bullet X)^\tau_s = (f \bullet X)_{s\wedge\tau} &= \sum_j \varphi_j\left(X_{t_{j+1}\wedge\tau\wedge s} - X_{t_j\wedge\tau\wedge s}\right) \\ &= \sum_j \varphi_j\left(X^\tau_{t_{j+1}\wedge s} - X^\tau_{t_j\wedge s}\right) \\ &= f \bullet (X^\tau)_s.\end{aligned}$$

(This calculation holds for *all* ST!) Since τ is discrete, we find

$$\begin{aligned}f(s)\mathbb{1}_{(0,\tau]}(s) &= \sum_j \varphi_j \mathbb{1}_{(0,\tau]}(s)\mathbb{1}_{(t_j,t_{j+1}]}(s) \\ &= \sum_j \varphi_j \mathbb{1}_{(\tau\wedge t_j,\tau\wedge t_{j+1}]}(s) \\ &= \sum_j \underbrace{\left(\varphi_j \mathbb{1}_{\{\tau>t_j\}}(s)\right)}_{\in\mathscr{F}_{t_j}} \underbrace{\mathbb{1}_{(t_j,t_{j+1}]}(s)}_{\tau\wedge t_{j+1}=t_{j+1}}.\end{aligned}$$

(In the last step, we used that $\tau(\omega) > t_j \implies \tau(\omega) \geqslant t_{j+1}$, discrete!). Hence, $f\mathbb{1}_{(0,\tau]} \in \mathscr{E}$ and

$$\{f\mathbb{1}_{(0,\tau]}\} X_s \bullet = \sum_j \varphi_j \mathbb{1}_{\{\tau > t_j\}}(s) \left(X_{t_{j+1}\wedge s} - X_{t_j \wedge s}\right)$$
$$= \sum_j \varphi_j \left(X_{t_{j+1}\wedge\tau\wedge s} - X_{t_j\wedge\tau\wedge s}\right)$$
$$= f \bullet (X^\tau)_s . \qquad \blacksquare$$

The crucial conclusion in Proposition 3.8 is that the formulae (3.8) and (3.9) induce an isometry between

$$\mathsf{L}^2(\mathbb{P})\text{-norm of} f \bullet X_t \quad \leftrightsquigarrow \quad \mathsf{L}^2(\langle X\rangle)\text{-norm of } f(t).$$

But the second expression is an «ordinary» Lebesgue integral, i.e. we can use (3.8) and (3.9) to extend the class of possible integrands. This is a standard technique from analysis that goes as follows:

$$\overline{\mathscr{E}} := \mathsf{L}^2(\langle X\rangle)\text{-completion of } \mathscr{E}$$
$$= \left\{ f \in \mathsf{L}^2(\langle X\rangle) : \quad \exists (f_n)_{n\in\mathbb{N}} \subset \mathscr{E},\ f_n \xrightarrow{\text{a.s.}} f \quad \text{and} \quad (f_n)_{n\in\mathbb{N}} \text{ is } \mathsf{L}^2(\langle X\rangle)\text{-Cauchy} \right\}.$$

Mind $\overline{E}$-functions are $\mathscr{P}$-measurable, because they are a.s. limits of $\mathscr{E}$-functions; but all functions in $\mathscr{E}$ are left continuous, i.e. $\mathscr{P}$-measurable!

Remark 3.9 (BLT principle). Extension by BLT = bounded linear transforms. Let therefore $f \in \mathscr{E}$ and $(f_n)_n \subset \mathscr{E}$ an approximating sequence, i.e.

$$\left\|f_n - f_m\right\|_{\mathsf{L}^2(\langle X\rangle)} = \mathbb{E}\int_0^\infty \left|f_n(u) - f_m(u)\right|^2 \mathrm{d}\langle X\rangle_u \xrightarrow{m,n\to\infty} 0$$
$$\left[\text{or } \mathbb{E}\int_0^\infty \left|f_n(u) - f(u)\right|^2 \mathrm{d}\langle X\rangle_u \xrightarrow{n\to\infty} 0\right]$$

But we know by the properties of the stochastic integral that for all $T \in [0,\infty]$

$$\mathbb{E}\left[\sup_{t\leqslant T}\left|f_n \bullet X_t - f_m \bullet X_t\right|^2\right] \overset{\text{Doob}}{=} 4\mathbb{E}\left|f_n \bullet X_T - f_m \bullet X_T\right|^2$$
$$\overset{3.8}{=} 4\mathbb{E}\int_0^T \left|f_n(u) - f_m(u)\right|^2 \mathrm{d}\langle X\rangle_u \xrightarrow{n,m\to\infty} 0.$$

Hence, $\sup_{t\leqslant T}\left|f_n \bullet X_t - f_m \bullet X_t\right|^2$ is a Cauchy sequence and we conclude that

$$\left(f_n \bullet X\right)_t \xrightarrow[n\to\infty]{\mathsf{L}^2(\mathbb{P}),\ \text{uniformly in } t\leqslant T} M_t$$

and the limit is again càdlàg and an L^2-MG. Since the conditional expectation is $\mathsf{L}^1(\mathbb{P})$-continuous — $\mathbb{E}\left|\mathbb{E}(Z \mid \mathscr{G})\right| \leqslant \mathbb{E}|Z|$ — we get

$$\mathbb{E}\Big(\left(M_t - M_s\right)^2 \,\Big|\, \mathscr{F}_s\Big) = \lim_{n\to\infty}\mathbb{E}\Big(\left(f_n \bullet X_t - f_n \bullet X_s\right)^1 \,\Big|\, \mathscr{F}_s\Big) \qquad (\star)$$
$$= \lim_{n\to\infty}\mathbb{E}\Big(\int_s^t f^2(u)\,\mathrm{d}\langle X\rangle_u \,\Big|\, \mathscr{F}_s\Big).$$

By Lemma 3.3, for $f \in \mathscr{E}$,

$$\langle M \rangle_t = \int_0^t f^2(u)\,\mathrm{d}\langle X \rangle_u \leqslant \int_0^\infty f^2(u)\,\mathrm{d}\langle X \rangle_u < \infty.$$

Important notice The just described procedure does not depend on the choice of the approximating sequence – cf. the step above marked by ($\star$): The RHS does not depend of the approximating sequence.

Definition 3.10. Let $f \in \mathscr{E}$ and $X \in \mathscr{M}^2$. Then we call

$$f \bullet X_t := \int_0^t f(s)\,\mathrm{d}X_s = \mathrm{L}^2(\mathbb{P}) - \lim_{n\to\infty} f_n \bullet X$$

the **stochastic integral of f with respect to X**. Hereby $(f_n)_{n\in\mathbb{N}} \subset \mathscr{E}$ is an arbitrary approximating sequence.

Proposition 3.11 (Properties of the Stochastic Integral). *Let $f \in \overline{\mathscr{E}}$ and $X \in \mathscr{M}^2$. Then*

(a) $f \mapsto f \bullet X : \overline{\mathscr{E}} \to \mathscr{M}^2$ *is linear.*

(b) $X \mapsto f \bullet X : \mathscr{M}^2 \to \mathscr{M}^2$ *is linear.*

(c) $f \bullet X \in \mathscr{M}^2$.

(d) $\langle f \bullet X \rangle_t = \int_0^t f^2(u)\,\mathrm{d}\langle X \rangle_u$.

(e) $\mathbb{E}\big[f \bullet X_t^2\big] = \mathbb{E}\int_0^t f^2(u)\,\mathrm{d}\langle X \rangle_u$.

(f) $\mathbb{E}\Big(\big(f \bullet X_t - f \bullet X_s\big)^2 \;\Big|\; \mathscr{F}_s\Big) = \mathbb{E}\int_t^s f^2(u)\,\mathrm{d}\langle X \rangle_u$.

(g) $t \mapsto f \bullet X_t$ *is càdlàg and* $\Delta f \bullet X_t = f(t-)\Delta X_t$.[a]

(h) $(f \bullet X)^\tau = f \bullet (X^\tau) = \big(f \mathbb{1}_{0,\tau]}\big)\,X$ *for all ST* τ.

[a] For a càdlàg process is $\Delta Y_t := Y_t - Y_{t-} = Y_t - \lim_{s\uparrow t} Y_s$.

Proof. The properties (a), (b) are evident, because the L^2-limit preserves linearity; (c) and the mapping properties in (a) and (b) follow from Remark 3.9; also (d), (e) and (f).

(g) The construction of the stochastic integral shows that $f_n \bullet X \xrightarrow{n\to\infty} f \bullet X$, where $f_n \in \mathscr{E}$, $f \in \overline{\mathscr{E}}$ and the convergence is in L^2 and uniform for $t \leqslant T$. Therefore, for a subsequence $\sup_{t\leqslant T}\left|f_{n_k} \bullet X_t - f \bullet X_t\right| \longrightarrow 0$ a.s., i.e. $f \bullet X$ inherits the càdlág paths of $f_{n_k} \bullet X$, cf. Proposition 3.8.

(h) The first equality holds for all τ and $f \in \mathscr{E}$, cf. Proposition 3.8. Let now $\mathscr{E} \ni f_n \xrightarrow{\mathrm{L}^2(\langle X \rangle)} f$. Since $\langle X \rangle$ is unique and by optional stopping $X \in \mathscr{M}^2 \implies X^\tau \in \mathscr{M}^2$, it follows that $\langle X^\tau \rangle = \langle X \rangle^\tau$ and in particular $\langle X^\tau \rangle_s \leqslant \langle X \rangle_s$.

By definition of the integral

$$\mathbb{E}\left[\sup_{t\geqslant 0}\left|f_n \bullet (X^\tau)_t - f \bullet (X^\tau)_t\right|^2\right] \leqslant 4\mathbb{E}\int_0^\infty \left|f_n(u) - f(u)\right|^2 \,\mathrm{d}\langle X\rangle_u^\tau$$
$$\leqslant 4\mathbb{E}\int_0^\infty \left|f_n(u) - f(u)\right|^2 \,\mathrm{d}\langle X\rangle_u \xrightarrow{n\to\infty} 0.$$

Easier to see is that $\left(f_n \bullet X\right)^\tau \xrightarrow{n\to\infty} (f \bullet X)^\tau$. Thus, the first equality follows by taking the limit via $\mathscr{E}$-integrands.

If τ is discrete and $f \in \overline{\mathscr{E}}$, then we find

$$\mathbb{E}\int_0^\infty \left|f(u)\mathbb{1}_{(0,\tau]}(u) - f_n(u)\mathbb{1}_{(0,\tau]}(u)\right|^2 \,\mathrm{d}\langle X\rangle_u \leqslant \mathbb{E}\int_0^\infty \left|f(u) - f_n(u)\right|^2 \,\mathrm{d}\langle X\rangle_u \xrightarrow{n\to\infty} 0.$$

(showing, in particular, that $f\mathbb{1}_{(0,\tau]} \in \overline{\mathscr{E}}$, i.e. the second equality holds for discrete τ by taking the limit with $\mathscr{E}$-integrads.

Finally, we approximate τ by discrete τ_n as in Proposition 2.1. So

$$\mathbb{E}\int_0^\infty \left|f(u)\mathbb{1}_{(0,\tau]}(u) - f(u)\mathbb{1}_{(0,\tau_n]}(u)\right|^2 \,\mathrm{d}\langle X\rangle_u \leqslant \mathbb{E}\int_0^\infty |f(u)|^2\, \mathbb{1}_{(\tau,\tau_n]} \,\mathrm{d}\langle X\rangle_u \xrightarrow{n\to\infty} 0,$$

and, since $f \in L^2(\langle X\rangle)$, the claim follows by dominated convergence using $\tau_n \to \tau$ and $\mathbb{1}_{(\tau,\tau_n]} \to 0$. ■

Now, we want to clarify two mainly technical questions that are important in understanding the theory.

(a) How **big** is $\overline{\mathscr{E}}$?

(b) Why have we constructed the stochastic integral as L^2-limit and not as Riemann integral (i.e. as an a.s. limit)?

How big is $\overline{\mathscr{E}}$?

Notation $\mathbb{L}$ denotes all *left continuous* processes with right limits.

Notation $\mathscr{B}(\mathscr{P})$ is the set of all $\mathscr{P}$-measurable processes,
$\mathscr{B}_b(\mathscr{P})$ the set of all bounded $\mathscr{P}$-measurable processes (recall Definition 1.15!).

Lemma 3.12. *The predictable σ-algebra $\mathscr{P}$ is generated by the family*

$$\mathscr{G} := \left\{(t,\infty)\times F \,:\, t\geqslant 0,\ F\in\mathscr{F}_t\right\}.$$

Proof. We have $\mathscr{G} \subset \mathscr{P}$, since $f(u,\omega) := \mathbb{1}_F(\omega)\mathbb{1}_{(t,\infty)}(u)$ with $F \in \mathscr{F}_t$ is adapted and left continuous.

Further, every $f \in \mathbb{L}$ can be approximated by a sequence $f_n \in \mathscr{E}$. Indeed (WLOG let $f(0) = 0$):

$$f_n(u,\omega) := \sum_{j=1}^{n2^n-1} \left[(-n)\vee f(j2^{-n},\omega)\wedge n\right]\mathbb{1}_{(j2^{-n},(j+1)2^{-n}]}(u) \xrightarrow[n\to\infty]{\text{a.s.}} f(u,\omega).$$

This means that $f \in \mathbb{L}$ is $\sigma(\mathscr{E})$-measurable. Since $\sigma(\mathbb{L})$ is the smallest σ-algebra with respect to which all functions of $\mathbb{L}$ are measurable, we get $\sigma(\mathbb{L}) \subset \sigma(\mathscr{E})$. Thus

$$\mathscr{P} = \sigma(\mathbb{L}) \subset \sigma(\mathscr{E}) \subset \sigma(\mathscr{G}).$$

By definition $\mathscr{G} \subset \mathscr{P}$ and so $\sigma(\mathscr{G}) \subset \sigma(\mathscr{P}) = \mathscr{P}$ the proof is finished. ■

Theorem 3.13. *Let* $X \in \mathscr{M}^2$. *Then*

$$\overline{\mathscr{E}} = \left\{ f \in \mathsf{L}^2(\langle X \rangle) : \quad f \in \mathscr{P} \right\}.$$

Proof. By definition, we have $\mathscr{E} \subset \mathsf{L}^2(\langle X \rangle) \cap \mathscr{B}_b(\mathscr{P})$. Thus

$$\overline{\mathscr{E}} \subset \mathsf{L}^2(\langle X \rangle) \cap \mathscr{B}(\mathscr{P}),$$

using that $\mathsf{L}^2(\langle X \rangle)$ is closed under $\mathsf{L}^2(\langle X \rangle)$-limits (which is an ordinary L^2-space, cf. Remark 3.9 (b)). Further, the limit is $\mathscr{P}$-measurable, because it can be identified by a.s. convergent subsequences.

For the converse, we require a so called **monotone class argument**: Consider therefore the set

$$\mathscr{D} := \left\{ f \in \mathscr{B}(\mathscr{P}) : \quad \forall \varepsilon > 0 \, \exists f_\varepsilon \in \mathscr{E} : \left\| f - f_\varepsilon \right\|_{\mathsf{L}^2(\langle X \rangle)} < \varepsilon \right\}.$$

Clearly, we see

(a) $\mathscr{D} \subset \overline{\mathscr{E}}$ – as $\mathscr{D}$ contains all processes approximated by $\mathscr{E}$;

(b) $\mathbb{1} \in \mathscr{D}$ and $\mathbb{1}_{(t,\infty)\times F} \in \mathscr{D}$ for $t \geqslant 0$, $F \in \mathscr{F}_t$;

(c) $\mathscr{D}$ is a vector space;

(d) If $f^n \in \mathscr{D}$ and $f^1 \leqslant f^2 \leqslant \ldots \leqslant f^n \leqslant \ldots$ and if $f := \sup_n f_n \in \mathsf{L}^\infty(\mathscr{P})$, then also $f \in \mathscr{D}$. Indeed: By dominated convergence, we get

$$\|f - f^n\|_{\mathsf{L}^2(\langle X \rangle)} < \frac{\varepsilon}{2} \qquad \forall n \geqslant N(\varepsilon)$$

und consequently

$$\exists f_{\varepsilon/2}^{N(\varepsilon)} \in \mathscr{S} \qquad \text{with} \qquad \left\| f^{N(\varepsilon)} - f_{\varepsilon/2}^{N(\varepsilon)} \right\|_{\mathsf{L}^2(\langle X \rangle)} < \frac{\varepsilon}{2}$$

and thus

$$\left\| f - f_{\varepsilon/2}^{N(\varepsilon)} \right\|_{\mathsf{L}^2(\langle X \rangle)} \leqslant \left\| f - f^{N(\varepsilon)} \right\|_{\mathsf{L}^2(\langle X \rangle)} + \left\| f^{N(\varepsilon)} - f_{\varepsilon/2}^{N(\varepsilon)} \right\|_{\mathsf{L}^2(\langle X \rangle)} < \varepsilon.$$

Hence, $f \in \mathscr{D}$.

Consider now $\mathscr{G} := \left\{ (t,\infty) \times F : t \geqslant 0, F \in \mathscr{F}_t \right\}$. Then it holds (cf. Lemma 3.12)

$$\mathscr{G} \subset \left\{ B \in \mathscr{P} : \mathbb{1}_B \in \mathscr{D} \right\}.$$

But the RHS is a Dynkin system by the defining properties of $\mathscr{D}$. As $\mathscr{G}$ is $\cap$-stable, we get

$$\mathscr{P} \overset{3.12}{=} \sigma(\mathscr{G}) = \delta(\mathscr{G}) \subset \left\{ B \in \mathscr{P} : \mathbb{1}_B \in \mathscr{D} \right\} \subset \mathscr{P}$$

and we find $B \in \mathscr{P} \iff \mathbb{1}_B \in \mathscr{D}$. By the Sombrero lemma follows

$$\mathscr{B}_b(\mathscr{P}) \subset \mathscr{D} \subset \overline{\mathscr{E}}.$$

Further,

$$\overline{\mathscr{B}_b(\mathscr{P})}^{\mathsf{L}^2(\langle X \rangle)} = \mathsf{L}^2(\langle X \rangle) \cap \mathscr{B}(\mathscr{P})$$

and we find

$$\mathsf{L}^2(\langle X \rangle) \cap \mathscr{B}(\mathscr{P}) \subset \overline{\mathscr{E}}.$$

■

Why does not «naïve» stochastic integration make sense?

Now, we show that the Riemann-Stieltjes ansatz cannot be successful in the definition of the stochastic integral! Therefore, we need a result from functional analysis.

Theorem 3.14. *Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space and $(\mathscr{Y}, |\cdot|)$ a normed linear space. Further let $(\Lambda_i)_{i\in I}$ a (arbitrary induced) family of linear operators $\Lambda_i : \mathscr{X} \to \mathscr{Y}$. Then it holds*

$$\sup_{i\in I} \left|\Lambda_i x\right| < \infty \qquad \forall x \in \mathscr{X} \qquad \Longrightarrow \qquad \sup_{i\in I}\ \sup_{\|x\|\leqslant 1} \left|\Lambda_i x\right| < \infty.$$

Let us consider the simple deterministic situation first: Let π_n be a dyadic partition of $[0,1]$, $\pi_n = \left\{j2^{-n} : j = 0, \dots, 2^n\right\}$. Then

$$\pi_1 \subset \pi_2 \subset \ldots \subset \pi_n \subset \ldots$$

and the refinement $\left|\pi_n\right| = 2^{-n} \to 0$. We are interested in in the Riemann-Stieltjes sums

$$S_n(f) = \sum_{t_j\in\pi_n} f(t_j)(x(t_{j+1}) - x(t_j)), \tag{3.10}$$

where $f, x : [0,1] \to \mathbb{R}$ and ask: When does $\lim_n S_n(f)$ exist?

Theorem 3.15. *Let $x : [0,1] \to \mathbb{R}$ be càdlàg. If $\lim_{n\to\infty} S_n(f) = S(f) < \infty$ for all $f \in \mathsf{C}_b[0,1]$, then $x : [0,1] \to \mathbb{R}$ is of bounded variation.*

Function analytic significance Theorem 3.15 gives a necessary (and by the following Theorem 3.16 also a sufficient) condition for the dual space of $\mathsf{C}[0,1]$: The dual is of the Form $\mathfrak{M}^+ - \mathfrak{M}^-$, where $\mathfrak{M}^+$ are the positive, finite measures on $[0,1]$, which can be identified with the probability distribution function with increasing functions on $[0,1]$.

Proof. We apply the Banach-Steinhaus theorem (Theorem 3.14) in the following situation: $\mathscr{X} = \mathsf{C}[0,1]$ with the supremum norm and $\mathscr{Y} = \mathbb{R}$ with absolute value. For $f \in \mathscr{X}$, let

$$\Lambda_n(f) := S_n(f) = \sum_{t_j\in\pi_n} f(t_j)(x(t_{j+1}) - x(t_j)).$$

For a fixed π_n there are $f_n \in \mathsf{C}[0,1]$ such that

$$\|f\|\infty = 1 \qquad \text{and} \qquad f_n(t_j) = \operatorname{sgn}(x(t_{j+1}) - x(t_j))$$

(«saw(tooth) wave» function). Then

$$
\begin{aligned}
& \Lambda_n(f_n) = \sum_{t_j \in \pi_n} \left| x(t_{j+1}) - x(t_j) \right| \\
\implies \quad & \sup_{\|f\|_\infty \leqslant 1} \left| \Lambda_n(f) \right| \geqslant \Lambda_n(f_n) = \sum_{t_j \in \pi_n} \left| x(t_{j+1}) - x(t_j) \right| \\
\implies \quad & \sup_n \sup_{\|f\|_\infty \leqslant 1} \left| \Lambda_n(f) \right| \geqslant \mathrm{var}(x; [0,1]).
\end{aligned}
$$

(Last step — exercise! Here var = variation). On the other hand, by assumption

$$
\begin{aligned}
& \lim_{n\to\infty} \Lambda_n(f) = S(f) < \infty && \forall f \in \mathfrak{X} \\
\implies \quad & \sup_n \left| \Lambda_n(f) \right| < \infty && \forall f \in \mathfrak{X} \\
\implies \quad & \sup_n \sup_{\|f\|_\infty} \left| \Lambda_n(f) \right| < \infty
\end{aligned}
$$

and hence $\mathrm{var}(x; [0,1]) < \infty$. ■

Therefore, we also need – necessarily – BV (= bounded variational) processes as integrators if we want to define the stochastic integral as such a limit.

Conversely,

Theorem 3.16. *Let f be left continuous with right limits (càglàd), x be increasing (or BV), both defined on* $[0,1]$. *Then the limit*

$$
\lim_{|\pi|\to 0} \sum_{t_j \in \pi_n} f(t_j)(x(t_{j+1}) - x(t_j)) = \int_0^t f(t) \mathrm{d}x(t)
$$

exists, where π are partitions of $[0,1]$ *with refinement* $|\pi|$.

Proof. Since f is càglàd, f has only finitely many jumps $|\Delta f| > \varepsilon$ in $[0,1]$, say at $\sigma = \{s_1, \dots, s_N\} \subset [0,1]$.[1] Therefore, there exists such a fine partition $\pi \supset \sigma$ such that

$$
\left| f(t) - f(t_j) \right| < 2\varepsilon \qquad \forall t_j \notin \sigma,\ t \in [t_j, t_{j+1}),
$$

and as x is càdlàg, π may be so fine that

$$
\left| x(t_j) - x(t_{j+1}) \right| < \varepsilon \qquad \forall t_j \in \pi.
$$

Next, let π' be another partition with $\left|\pi'\right| < |\pi|$. WLOG we assume $\pi' \supset \pi$, else we consider the partition sums to π and $\pi' \cup \pi$, resp. to π' and $\pi' \cup \pi$, in the calculations

[1] If not there is an accumulation point of e.g. jumps $\Delta f > \varepsilon$, which would make f unbounded.

below.

$$
\begin{aligned}
&\left|\sum_{t_j\in\pi} f(t_j)(x(t_{j+1})-x(t_j)) - \sum_{u_j\in\pi'} f(u_k)(x(u_{k+1})-x(u_k))\right|\\
&\quad= \left|\sum_{t_j\in\pi}\ \sum_{u_k\in(t_j,t_{j+1})\cap\pi'} (f(t_j)-f(u_k))(x(u_{k+1})-x(u_k))\right|\\
&\quad= \left|\sum_{t_j\in\pi}\ \sum_{\substack{u_k\in(t_j,t_{j+1})\cap\pi'\\ [t_j,u_k)\cap\sigma=\emptyset}} \dots + \sum_{t_j\in\pi}\ \sum_{\substack{u_k\in(t_j,t_{j+1})\cap\pi'\\ [t_j,u_k)\cap\sigma\neq\emptyset}} \dots\right|\\
&\quad\leqslant \sum_{t_j\in\pi\setminus\sigma}\ \sup_{u\in(t_j,t_{j+1}]} \left|f(t_j)-f(u)\right|\left|x(t_{j+1}-x(t_j)\right| + 2\,\|f\|_\infty \sum_{t_j\in\pi}\ \sum_{\substack{u_k\in(t_j,t_{j+1})\cap\pi'\\ [t_j,u_k)\cap\sigma\neq\emptyset}} (x(u_{k+1})-x(u_k))\\
&\quad\leqslant 2\varepsilon\,\mathrm{var}(x;[0,1]) + 2\,\|f\|_\infty\, N\, c(\left|\pi'\right|),
\end{aligned}
$$

where $c(\left|\pi'\right|) \xrightarrow{|\pi'\to 0|} 0$ by right continuity of x and since $\left|x(t_j)-x(t_{j+1}-)\right| < \varepsilon$ follows $\left|x(u_{k+1})-x(u_k)\right| \xrightarrow{|\pi'\to 0|} 0.$

Hence, the partition sums are a Cauchy sequence and thus converge. ∎

Sadly, only a few interesting stochastic processes are actually BV.

Example 3.17. An LP(d) with Lévy triple $(\ell, 0, \nu)$ is BV if and only if $\int_{|x|\leqslant 1} |x|\ \nu(\mathrm{d}x) < \infty$. Cf. LÉVY, Chapter 9.

Example 3.18. Let $(B_t)_{t\geqslant 0}$ be a one-dimensional BM, $\pi_n = \left\{t\,j2^{-n} : j = 0,\dots,2^n\right\}$ and $t \geqslant 0$. Then (exercise!)

$$
\mathrm{L}^2 - \lim_{n\to\infty} \sum_{t_j\in\pi_n} \left(B_{t_{j+1}} - B_{t_j}\right)^2 = t.
$$

Idea: Calculate

$$
\mathbb{E}\left|\sum_{t_j\in\pi_n} \left(B_{t_{j+1}} - B_{t_j}\right)^2 - (t_{j+1}-t_j)^2\right|^2.
$$

Note: $\langle B\rangle_t = t$ — therefore «quadratic variation».

Example 3.19. A BM $(B_t)_{t\geqslant 0}$ has unbounded variation on $[0,t]$.

Indeed: Using again the notation as in Example 3.18, by continuity of the Brownian paths, we have

$$\left|B_{t_{j+1}} - B_{t_j}\right| \leqslant \varepsilon \qquad \forall t_j \in \pi_n \forall n \geqslant N(\varepsilon) \gg 1$$

and thus there is a subsequence and a.s. convergence:

$$\begin{aligned} t &\overset{\text{a.s.}}{=} \lim_{n(k)\to\infty} \sum_{t_j\in\pi_{n(k)}} \left(B_{t_{j+1}} - B_{t_j}\right)^2 \\ &\leqslant \varepsilon \limsup_{n(k)\to\infty} \sum_{t_j\in\pi_{n(k)}} \left|B_{t_{j+1}} - B_{t_j}\right| \\ &\leqslant \varepsilon \ \operatorname{var}(\mathsf{B};[0,t]). \end{aligned}$$

Since $t > 0$, letting $\varepsilon \to 0$, it follows necessarily that $\operatorname{var}(\mathsf{B};[0,t]) = \infty$.

Chapter 4

Extension of the Integral

Next, we want to consider stochastic integrals which are driven by local L^2-MG. We commence with an auxiliary statement first.

Lemma 4.1. *Let $X \in \mathcal{M}^2_{\mathrm{loc}}$ a local L^2-MG. Then there is an $\mathcal{P}$-measurable increasing process $\langle X\rangle$ such that $(X_t^2 - \langle X\rangle_t)_{t\geqslant 0}$ is a local MG.*

Proof. Let $(T_n)_n$ be a reducing sequence for $X \in \mathcal{M}^2_{\mathrm{loc}}$, i.e. $T_n \xrightarrow{\mathbb{P}} \infty$ and $X^{T_n} \in \mathcal{M}^2 \quad \forall n$. Hence

$$\Longrightarrow \qquad (X_t^{T_n})^2 - \left\langle X^{T_n}\right\rangle_t \quad \text{is a MG} \quad \forall n.$$

Moreover,

$$X^{T_n}_{t\wedge T_m} = X_t^{T_n\wedge T_m} = X_t^{T_m} \qquad \forall m < n$$

$$\overset{\text{Doob-Meyer}}{\underset{\text{unique}}{\Longrightarrow}} \quad \left\langle X^{T_n}\right\rangle_{t\wedge T_m} = \left\langle X^{T_m}\right\rangle_t \qquad \forall m < n$$

i.e. the process

$$\langle X\rangle_t := \left\langle X^{T_n}\right\rangle_t, \qquad t \leqslant T_n,$$

is well-defined, $\mathcal{P}$-measurable (exercise!) and increasing. Further,

$$X^2_{t\wedge T_n} - \langle X\rangle_{t\wedge T_n} = (X_t^{T_n})^2 - \left\langle X^{T_n}\right\rangle_t$$

is a MG, i.e. $X^2 - \langle X\rangle$ is a local MG. ■

We now want to define

$$\int f(s)\,\mathrm{d}X_s \qquad \text{for} \qquad f \in \mathrm{L}^2_{\mathrm{loc}}(\langle X\rangle) \cap \mathcal{B}(\mathcal{P}),\ X \in \mathcal{M}^2_{\mathrm{loc}}. \tag{4.1}$$

Note that

$$f \in \mathrm{L}^2_{\mathrm{loc}}(\langle X\rangle) \iff f^{\tau_n} \in \mathrm{L}^2(\langle X\rangle) \iff \mathbb{E}\int_0^{\tau_n} |f(s)|^2\,\mathrm{d}\langle X\rangle_s < \infty$$

for an reducing sequence $(\tau_n)_{n\in\mathbb{N}}$. Let $(T_n)_n$ be a reducing sequence for X. Set

$$S_n := \tau_n \wedge T_n.$$

Then S_n is a reducing sequence for f and X.

Set $f_n(s,\omega) := f(s,\omega)\mathbb{1}_{\{s\leqslant S_n\}}(s,\omega)$. Since $\mathbb{1}_{\{s\leqslant S_n\}}(s,\omega) = \mathbb{1}_{[0,S_n(\omega)]}(s)$ is left continuous, so $f_n \in \mathscr{P}$ and

$$\mathbb{E}\int_0^\infty \left|f_n(s)\right|^2 \,\mathrm{d}\left\langle X^{T_n}\right\rangle_s = \mathbb{E}\int_0^{S_n} |f(s)|^2 \,\mathrm{d}\left\langle X^{T_n}\right\rangle_s = \mathbb{E}\int_0^{S_n} |f(s)|^2 \,\mathrm{d}\langle X\rangle_{s\wedge T_n} = \mathbb{E}\int_0^{S_n} |f(s)|^2 \,\mathrm{d}\langle X\rangle_s < \infty,$$

as $f^{\tau_n} \in \mathrm{L}^2(\langle X\rangle)$ and $S_n \leqslant \tau_n$.

Therefore, the stochastic integrals

$$M_t^n := \int_0^t f_n(s)\,\mathrm{d}X_s^{T_n} \in \mathscr{M}^2 \qquad \forall n$$

are defined and it holds

$$\forall m < n: \qquad M_{t\wedge S_m}^n = \int_0^{t\wedge S_m} f_n(s)\,\mathrm{d}X_s^{T_n} \overset{\text{def}}{\underset{f_m}{=}} \int_0^t f_m(s)\,\mathrm{d}X_s^{T_n} = \int_0^t f_m(s)\,\mathrm{d}X_s^{T_m} = M_t^m.$$

Thus exists $\mathsf{M} \in \mathscr{M}^2_{\mathrm{loc}}$ defined by

$$M_t := M_t^n \quad \text{for} \quad t \leqslant S_n, \qquad M_t^{S_n} = M_t^n.$$

Definition 4.2. Let $\mathsf{X} \in \mathscr{M}^2_{\mathrm{loc}}$ and $f \in \mathrm{L}^2_{\mathrm{loc}}(\langle X\rangle)$. The uniquely determined element $\mathsf{M} \in \mathscr{M}^2_{\mathrm{loc}}$, described by the procedure above, is called the **stochastic integral of f with respect to** X, and is denoted by

$$\mathsf{M} = f \bullet \mathsf{X} = \int_0^\bullet f(s)\,\mathrm{d}X_s.$$

Let us discuss an interesting special case, simplifying the structure of $\mathrm{L}^2_{\mathrm{loc}}(\langle X\rangle)$.

Lemma 4.3. *Let* $\mathsf{X} \in \mathscr{M}^2_{\mathrm{loc}}$ *such that* $t \mapsto \langle X\rangle_t$ *is continuous. Then*

$$f \in \mathrm{L}^2_{\mathrm{loc}}(\langle X\rangle) \iff \int_0^t |f(s,\omega)|^2 \,\mathrm{d}\langle X(\omega)\rangle_s < \infty \qquad \text{a.s.} \quad \forall t.$$

Proof. $\Longrightarrow$ Let $f \in \mathrm{L}^2_{\mathrm{loc}}(\langle X\rangle)$ with reducing sequence τ_n. Then

$$\mathbb{E}\int_0^{\tau_n} |f(s)|^2 \,\mathrm{d}\langle X\rangle_s < \infty \implies \int_0^{\tau_n(\omega)} |f(s,\omega)|^2 \,\mathrm{d}\langle X(\omega)\rangle_s < \infty$$

a.s. for all n. Since $\tau_n \to \infty$, we have for every fixed ω that $\tau_n(\omega) \geqslant t$ for $n \geqslant n_t \gg 1$

$\Longleftarrow$ Conversely, set

$$\tau_n := \inf\left\{t > 0 : \int_0^t |f(s,\omega)|^2 \,\mathrm{d}\langle X(\omega)\rangle_s \geqslant n\right\}.$$

Then τ_n is a ST and, by continuity of the angle bracket, it holds that

$$\mathbb{E}\int_0^{\tau_n} |f(s)|^2 \, \mathrm{d}\langle X\rangle_s \leqslant \mathbb{E}n = n < \infty. \qquad \blacksquare$$

Finally, we define the stochastic integral.

Definition 4.4. A **semimartingale (SMG)** with respect to the filtration $(\mathscr{F}_t)_{t\geqslant 0}$ is a stochastic process of the form

$$X_t = M_t + A_t, \tag{4.2}$$

where $\mathsf{M} \in \mathscr{M}^2_{\mathrm{loc}}$ (wrt. $(\mathscr{F}_t)_{t\geqslant 0}$) and $\mathsf{A} \in \mathrm{BV} \cap \mathscr{P}$.

Notation

$$\begin{aligned} |A|_t &:= \mathrm{var}(\mathsf{A};[0,t]) \qquad \text{total variation of A}, \\ \mathsf{L}^1(|A|) &:= \left\{ f : \int_0^\infty |f(s,\cdot)| \, \mathrm{d}|A|_s < \infty \quad \text{a.s.} \right\}, \\ f \in \mathsf{L}^1_{\mathrm{loc}}(|A|) &\iff f^{\tau_n} \in \mathsf{L}^1_{\mathrm{loc}}(|A|) \quad \text{for a reducing sequence } (\tau_n)_n. \end{aligned} \tag{4.3}$$

Definition 4.5. Let $X_t = M_t + A_t$ be a SMG and we assume

$$f \in \mathsf{L}^2_{\mathrm{loc}}(\langle X\rangle) \cap \mathsf{L}^1_{\mathrm{loc}}(|A|).$$

Then the **stochastic integral of f with respect to** X is given by

$$f \bullet X_t = \int_0^t f(s)\,\mathrm{d}X_s := \int_0^t f(s)\,\mathrm{d}M_s + \int_0^t f(s)\,\mathrm{d}A_s, \tag{4.4}$$

where the second integral is defined in the Riemann-/Lebesgue-Stieltjes sense.

Mind $f \bullet \mathsf{X}$ is again a SMG! Indeed: $f \bullet \mathsf{M} \in \mathscr{M}^2_{\mathrm{loc}}$ by Definition 4.2, whereas $t \mapsto \int_0^t f(s)\,\mathrm{d}A_s$ is clearly BV.

Obviously we have not answered yet the question if (4.4) is well-defined. The proof in the general case is complicated, cf. [Pro05, § 2, Theorem 2], whereby the decomposition $X_t = M_t + A_t$, with $\mathsf{M} \in \mathscr{M}_{\mathrm{loc}}$ and $\mathsf{A} \in \mathscr{P} \cap \mathrm{BV}$, is unique. Here, we only deal with two special cases that are sufficient in general.

Proposition 4.6. *Let $X_t = X_t + A_t$ with $\mathsf{M} \in \mathscr{M}^c_{\mathrm{loc}}$ (continuous local MG) and* A *a* BV *process. Then the representation of* X *is unique (up to constants).*

Proof. Let $M_t + A_t = M'_t + A'_t$ with $\mathsf{M}, \mathsf{M}' \in \mathscr{M}^c_{\mathrm{loc}}$ and A, A' be BV processes. Then

$$M_t - M'_t = A'_t - A_t \qquad \text{is} \qquad \mathrm{BV} \cap \mathscr{M}^c_{\mathrm{loc}}.$$

For a reducing sequence $S_\ell := T_\ell \wedge T'_\ell$, we have

$$N_t = (M_t - M'_t)^{S_\ell} = (A'_t - A_t)^{S_\ell} = C_t \qquad \text{is} \qquad \mathrm{BV} \cap \mathscr{M}^c.$$

In particular, for $0 = t_0 < t_1 < \ldots < t_n = t$,

$$\mathbb{E}(N_t - N_0)^2 = \mathbb{E}\left[\sum_{j=0}^{n-1}\left((N_{t_{j+1}} - N_0)^2 - (N_{t_j} - N_0)^2\right)\right]$$

$$\overset{\text{MG}}{\underset{3.2}{=}} \sum_{j=0}^{n-1}\mathbb{E}\left(N_{t_{j+1}} - N_{t_j}\right)^2.$$

Set $V_t := \operatorname{var}(\mathsf{N};[0,t])$ and $\sigma_k := \inf\left\{t > 0 : V_t \geqslant k\right\}$. Then σ_k is a ST and we find

$$\begin{aligned}
\mathbb{E}(N_{t\wedge\sigma_k} - N_0)^2 &= \sum_{j=0}^{n-1}\mathbb{E}\left[N^{\sigma_k}_{t_{j+1}} - N^{\sigma_k}_{t_j}\right]^2 \\
&\leqslant \sum_{j=0}^{n-1}\mathbb{E}\left[\sup_\ell\left|N^{\sigma_k}_{t_{l+1}} - N^{\sigma_k}_{t_l}\right|\left(N^{\sigma_k}_{t_{j+1}} - N^{\sigma_k}_{t_j}\right)\right] \\
&= \mathbb{E}\left[\sup_\ell\left|N^{\sigma_k}_{t_{l+1}} - N^{\sigma_k}_{t_l}\right|\underbrace{\sum_{j=0}^{n-1}\left(N^{\sigma_k}_{t_{j+1}} - N^{\sigma_k}_{t_j}\right)}_{\leqslant V_{t\wedge\sigma_k}\leqslant k}\right] \\
&\leqslant k\,\mathbb{E}\sup_\ell\left|N^{\sigma_k}_{t_{l+1}} - N^{\sigma_k}_{t_l}\right| \xrightarrow{n\to\infty} 0,
\end{aligned}$$

since the process N_ℓ is continuous and $\left|N^{\sigma_k}_{t_{l+1}} - N^{\sigma_k}_{t_l}\right| \leqslant k$ (so we can use dominated convergence).

Further, $\sigma_k, S_\ell \xrightarrow[\ell\to\infty]{k\to\infty} \infty$ a.s. (if necessary for a subsequences). Hence, it follows that $N_t \equiv 0$ oder $M_t - M_0 = M'_t - M'_0$. ■

Corollary 4.7. *It holds* $\mathrm{BV} \cap \mathscr{M}^2_{\mathrm{c}} = \left\{(X_t)_{t\geqslant 0} : X_t \equiv X_0\right\}$.

The case interesting for us follows from the next result that is interesting in itself.

Theorem 4.8. *Let* $(\pi_k)_k$ *be a sequence of partitions of* $[0,T]$ *such that* $\left|\pi_k\right| \to 0$. *Let* f *be càdlàg,* $\mathscr{F}_t$*-adapted and* $\mathsf{M} \in \mathscr{M}^2_{\mathrm{loc}}$. *Then the Riemann-Stieltjes sum*

$$\forall t \leqslant T: \qquad Y^{\pi_k}_t := \sum_{t_j\in\pi_k} f(t_j\wedge t)\left(M_{t_{j+1}\wedge t} - M_{t_j\wedge t}\right) \tag{4.5}$$

converges uniformly in probability (ucp) *(on compact t-sets) to* $\int_0^t f(s-)\,\mathrm{d}M_s$, *i.e.*

$$\forall \varepsilon > 0: \qquad \lim_{n\to\infty}\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\left|Y^{\pi_n}_t - \int_0^t f(s-)\,\mathrm{d}M_s\right| > \varepsilon\right) = 0. \tag{4.6}$$

Notation For f and a partition π we write

$$f^\pi(s) = \sum_{t_j\in\pi} f(t_j)\mathbb{1}_{[t_j,t_{j+1})}(s).$$

Clearly then $Y^\pi = \int_0^\bullet f^\pi(s-)\,\mathrm{d}M_s$.

Proof. Note first that $s \mapsto f(s-)$ is left continuous and thus $\mathscr{P}$-measurable, and that f is bounded on $[0,n]$, for $n > 0$. Set

$$\sigma_n := \inf\left\{s > 0 : |f(s)|^2 > n\right\}.$$

Then σ_n is a ST and $\sigma_n \to \infty$ a.s. For every reducing sequence $(\tau_n)_n$ of $\mathrm{M} \in \mathscr{M}^2_{\mathrm{loc}}$

$$\mathbb{E}\int_0^{\sigma_n\wedge\tau_n} f^2(s-)\,\mathrm{d}\langle M\rangle_s \leqslant \mathbb{E}\int_0^{\sigma_n\wedge\tau_n} \sup_{s\in[0,\sigma_n]} \left|f^2(s-)\right|\,\mathrm{d}\langle M\rangle_s$$
$$\leqslant n\,\mathbb{E}\langle M\rangle_{\tau_n} < \infty.$$

Hence $\int_0^t f(s-)\,\mathrm{d}M_s \in \mathscr{M}^2_{\mathrm{loc}}$ is well-defined and $\tau_n \wedge \sigma_n$ is a reducing sequence for M and $f \bullet \mathrm{M}$.

For $T > 0$, by Doob's inequality and by $\left\langle M^{\sigma_n\wedge\tau_n}\right\rangle_t = \langle M\rangle_{\sigma_n\wedge\tau_n\wedge t}$, we have

$$\mathbb{E}\left[\sup_{t\leqslant T}\left|Y^{\pi_k}_{t\wedge\sigma_n\wedge\tau_n} - \int_0^{t\wedge\sigma_n\wedge\tau_n} f(s-)\,\mathrm{d}M_s\right|^2\right]$$
$$\leqslant 4\mathbb{E}\int_0^{T\wedge\sigma_n\wedge\tau_n} |f^{\pi_k}(s-) - f(s-)|^2\,\mathrm{d}\langle M\rangle_s \xrightarrow[\text{(DOM)}]{k\to\infty} 0.$$

(Note: $f^2(s-) \leqslant n$ for $s \leqslant \sigma_n$). Here

$$f^{\pi_k}(s) := \sum_{t_j\in\pi_k} f(t_j)\mathbb{1}_{[t_j,t_{j+1})}(s)$$

Since $\sigma_n \wedge \tau_n \xrightarrow[n\to\infty]{\mathbb{P}} \infty$, also $\mathbb{P}\big(\sigma_n \wedge \tau_n < T\big) \xrightarrow{n\to\infty} 0$, i.e.

$$\forall\,\delta > 0\,\exists N_\delta\,\forall n \geqslant N_\delta : \qquad \mathbb{P}\big(\sigma_n \wedge \tau_n < T\big) < \delta.$$

Consequently, for fixed $T > 0$,

$$\mathbb{P}\left(\sup_{t\leqslant T}\left|Y^{\pi_k}_{t\wedge\sigma_n\wedge\tau_n} - \int_0^{t\wedge\sigma_n\wedge\tau_n} f(s-)\,\mathrm{d}M_s\right| > \varepsilon\right)$$
$$\leqslant \mathbb{P}\left(\sup_{t\leqslant T}\left|Y^{\pi_k}_{t\wedge\sigma_n\wedge\tau_n} - \int_0^{t\wedge\sigma_n\wedge\tau_n} f(s-)\,\mathrm{d}M_s\right| > \varepsilon, \sigma_n\wedge\tau_n \geqslant T\right) + \mathbb{P}\big(\sigma_n\wedge\tau_n < T\big)$$
$$\overset{\text{Chebyshev}}{\leqslant} \frac{4}{\varepsilon^2}\mathbb{E}\int_0^{T\wedge\sigma_n\wedge\tau_n} |f^{\pi_k}(s-) - f(s-)|\,\mathrm{d}\langle M\rangle_s + \delta$$
$$\xrightarrow{k\to\infty} \delta \xrightarrow{\delta\to 0} 0.$$

■

Corollary 4.9. *In the situation of Theorem 4.8 (in particular f is càdlàg) the Definition 4.5 of the stochastic integral is independent of the decomposition of the SMG* X.

Proof. Let $\mathrm{X} = \mathrm{M} + \mathrm{A} = \mathrm{M}' + \mathrm{A}'$ with $\mathrm{M}, \mathrm{M}' \in \mathscr{M}^2_{\mathrm{loc}}$ and $\mathrm{A}, \mathrm{A}' \in \mathrm{BV}$. Then

$$\underbrace{\int_0^\bullet f(s-)\,\mathrm{d}(M - M')_s}_{\text{Itô \& Stieltjes-}\int\text{, ucp}} = \underbrace{\int_0^\bullet f(s-)\,\mathrm{d}(A' - A)_s}_{\text{Stieltjes-}\int}.$$

Because the Stieltjes integral converges for almost all ω, we also have ucp-convergence and for the random variable holds

$$\int_0^t f(s-)\,\mathrm{d}M_s - \int_0^t f(s-)\,\mathrm{d}M'_s = \int_0^t f(s-)\,\mathrm{d}A_s - \int_0^t f(s-)\,\mathrm{d}A'_s.$$

The well-definedness of (4.4) for càdlàg integrands $f(s,\omega)$ follows. ■

Corollary 4.10. *The claim of Theorem 4.8 also holds for SMG of the form* $\mathsf{X} = \mathsf{M} + \mathsf{A}$, *where* $\mathsf{M} \in \mathscr{M}^2_{\mathrm{loc}}$ *and* $\mathsf{A} \in \mathrm{BV}$.

Proof. By Theorem 4.8, locally uniformly (in t) for almost all ω, it holds

$$\int_0^t f^{\pi_k}(s-,\omega)\,\mathrm{d}A_s(\omega) \xrightarrow{|\pi_k|\to 0} \int_0^t f(s-,\omega)\,\mathrm{d}A_s(\omega)$$

und thus follows ucp-convergence. By (exercise!)

$$\mathsf{Y}^n \xrightarrow{\text{ucp}} Y, \quad \mathsf{Z}^n \xrightarrow{\text{ucp}} Z \qquad \Longrightarrow \qquad \mathsf{Y}^n + \mathsf{Z}^n \xrightarrow{\text{ucp}} \mathsf{Y} + \mathsf{Z},$$

hence follows the claim. ■

Corollary 4.11. *Let* $\mathsf{X} = \mathsf{M} + \mathsf{A}$ *be a SMG and* $f \in \mathrm{L}^2_{\mathrm{loc}}(\langle X\rangle) \cap \mathrm{L}^1_{\mathrm{loc}}(|A|) \cap \mathscr{B}(\mathscr{P})$. *Further, let* $(f_k)_k \subset \mathscr{B}_b(\mathscr{P})$ *such that for every reducing sequence* $(\tau_n)_{n\in\mathbb{N}}$ *and all* $n \in \mathbb{N}$:

$$\lim_{n\to\infty}\left[\mathbb{E}\int_0^{\tau_n} \left|f(s) - f_k(s)\right|^2\,\mathrm{d}\langle M\rangle_s + \int_0^{\tau_n} \left|f(s) - f_k(s)\right|\,\mathrm{d}A_s\right] = 0.$$

Then $f_k \bullet \mathsf{X} \xrightarrow{\text{ucp}} f \bullet \mathsf{X}$.

Proof. For fixed ω and $T > 0$, we have $\tau_n(\omega) \geqslant T$, if $n \gg 1$. Therefore,

$$\sup_{t\leqslant T}\left|\int_0^t f(s,\omega)\,\mathrm{d}A_s(\omega) - \int_0^t f_k(s,\omega)\,\mathrm{d}A_s(\omega)\right| \leqslant \int_0^{\tau_n(\omega)} \left|f(s,\omega) - f_k(s,\omega)\right|\,\mathrm{d}\,|A|_s\,(\omega) \xrightarrow{k\to\infty} 0.$$

Thus it suffices to consider the MG case. WLOG let $(\tau_n)_n$ be also a reducing sequence for M. Then

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant T}\left|f_k \bullet M_t^{\tau_n} - f \bullet M_t^{\tau_n}\right|^2\right] &\overset{\text{Doob}}{\leqslant} 4\mathbb{E}\int_0^{T\wedge\tau_n} \left|f_k(s) - f(s)\right|^2\,\mathrm{d}\langle M\rangle_s \\ &\leqslant 4\mathbb{E}\int_0^{\tau_n} \left|f_k(s) - f(s)\right|^2\,\mathrm{d}\langle M\rangle_s\,.\end{aligned}$$

As in the proof of Theorem 4.8, finally, for all $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{t\leqslant T}\left|f_k \bullet M_t - f \bullet M_t\right| > \varepsilon\right) &\leqslant \mathbb{P}\left(\sup_{t\leqslant T}\left|f_k \bullet M_t - f \bullet M_t\right| > \varepsilon, \tau_n > t\right) + \mathbb{P}\big(\tau_n \leqslant t\big) \\ &\leqslant \frac{4}{\varepsilon^2}\mathbb{E}\int_0^{\tau_n} \left|f_k(s) - f(s)\right|^2\,\mathrm{d}\langle M\rangle_s + \mathbb{P}\big(\tau_n \leqslant t\big) \\ &\xrightarrow{k\to\infty} \mathbb{P}\big(\tau_n \leqslant t\big) \xrightarrow{n\to\infty} 0.\end{aligned}$$

■

Chapter 5

Orthogonal Martingales

We now show that the angle bracket $\langle X \rangle$ defines a geometric structure on $\mathscr{M}^2$.

Definition 5.1. Let $X, Y \in \mathscr{M}^2$. Then we call

$$\langle X, Y \rangle_t := \frac{1}{4} \left(\langle X + Y \rangle_t - \langle X - Y \rangle_t \right) \tag{5.1}$$

the **(predictable) quadratic covariation** or **angle bracket**.

Clearly: $\langle X \rangle = \langle X, X \rangle$.

Lemma 5.2. *Let $X, Y \in \mathscr{M}^2$. Then $X_t Y_t - \langle X, Y \rangle_t$ is a MG and*

$$\forall s \leqslant t : \qquad \mathbb{E}\big((X_t - X_s)(Y_t - Y_s) \bigm| \mathscr{F}_s \big) = \mathbb{E}\big(\langle X, Y \rangle_t - \langle X, Y \rangle_s \bigm| \mathscr{F}_s \big). \tag{5.2}$$

Proof. We calculate

$$\begin{aligned} 4X_t Y_t - 4\langle X, Y \rangle_t &= (X_t + Y_t)^2 - (X_t - Y_t)^2 - \langle X + Y \rangle_t - \langle X - Y \rangle_t \\ &\in \mathscr{M} - \mathscr{M} \subset \mathscr{M}. \end{aligned}$$

Analogous we see (5.2) by means of Remark 3.2 (c). ■

Our first goal is to characterise the stochastic integral by means of the angle bracket in an elegant way. Therefore, we prove a Cauchy-Schwarz-type inequality for $\langle \bullet, \bullet \rangle$.

Lemma 5.3 (Kunita-Watanabe inequalities). *Let $X, Y \in \mathscr{M}^2$ and $f \in L^2_{\text{loc}}(\langle X \rangle)$, $g \in L^2(\langle Y \rangle)$. Then we have $\int_0^{\tau_k} |f(s)g(s)| \,\mathrm{d}\,|\langle X, Y \rangle|_s < \infty$ for a reducing sequence $(\tau_k)_k$ and*

$$\int_0^{\tau_k} |f(s)g(s)| \,\mathrm{d}\,|\langle X, Y \rangle|_s \leqslant \sqrt{\int_0^{\tau_k} |f(s)|^2 \,\mathrm{d}\,\langle X \rangle_s} \sqrt{\int_0^{\tau_k} |g(s)|^2 \,\mathrm{d}\,\langle Y \rangle_s}, \tag{5.3}$$

$$\left| \int_0^{\tau_k} f(s)g(s) \,\mathrm{d}\,\langle X, Y \rangle_s \right| \leqslant \sqrt{\int_0^{\tau_k} |f(s)|^2 \,\mathrm{d}\,\langle X \rangle_s} \sqrt{\int_0^{\tau_k} |g(s)|^2 \,\mathrm{d}\,\langle Y \rangle_s}. \tag{5.4}$$

Proof. WLOG let $(\tau_k)_k$ be a reducing sequence for f and g, and WLOG let $\tau_k \xrightarrow[k \to \infty]{\text{a.s.}} \infty$. For $s < t$,

$$\langle X, Y \rangle_s^t := \langle X, Y \rangle_t - \langle X, Y \rangle_s$$

is a positive definite bilinear form (exercise!), i.e. we have the usual Cauchy-Schwarz inequality

$$\left| \langle X, Y \rangle_t - \langle X, Y \rangle_s \right| \leqslant \sqrt{\langle X \rangle_t - \langle X \rangle_s} \sqrt{\langle Y \rangle_t - \langle Y \rangle_s}, \tag{5.5}$$

$$\operatorname{var}\left(\langle X, Y \rangle, [s, t] \right) \overset{\text{ex!}}{\leqslant} \sqrt{\langle X \rangle_t - \langle X \rangle_s} \sqrt{\langle Y \rangle_t - \langle Y \rangle_s}. \tag{5.6}$$

Now, let $f, g \in \mathscr{E}$ with representation

$$f = \sum_{\text{finite}} \varphi_j \mathbb{1}_{(s_j, s_{j+1}]} \qquad \text{and} \qquad g = \sum_{\text{finite}} \gamma_j \mathbb{1}_{(s_j, s_{j+1}]},$$

where $0 = s_0 < s_1 < \ldots < s_n < \ldots$ and $\varphi_j, \gamma_j \in \mathscr{B}(\mathscr{F}_{s_j})$. To simplify notation, we set $s'_j := s_j \wedge \tau_k$. Then

$$\begin{aligned}
\left|\int_0^{\tau_k} f g \,\mathrm{d}\langle X, Y\rangle_s\right| &= \left|\sum_{j=0}^{n-1} \varphi_j \gamma_j \left(\langle X, Y\rangle_{s'_{j+1}} - \langle X, Y\rangle_{s'_j}\right)\right| \\
&\leqslant \sum_{j=0}^{n-1} \left|\varphi_j \gamma_j\right| \left|\langle X, Y\rangle_{s'_{j+1}} - \langle X, Y\rangle_{s'_j}\right| \\
&\leqslant \sum_{j=0}^{n-1} \left|\varphi_j \gamma_j\right| \operatorname{var}(\langle X, Y\rangle, [s'_j, s'_{j+1}]) \\
&\left[= \int_0^{\tau_k} |f g| \,\mathrm{d}\,|\langle X, Y\rangle|_s\right] \\
&\overset{(5.6)}{\leqslant} |\varphi_j| \sqrt{\langle X\rangle_{s'_{j+1}} - \langle X\rangle_{s'_j}}\, |\gamma_j| \sqrt{\langle Y\rangle_{s'_{j+1}} - \langle Y\rangle_{s'_j}} \\
&\overset{\text{CSI}}{\leqslant} \sqrt{\sum_{j=0}^{n-1} |\varphi_j|^2 \left(\langle X\rangle_{s'_{j+1}} - \langle X\rangle_{s'_j}\right)} \sqrt{\sum_{j=0}^{n-1} |\gamma_j|^2 \left(\langle Y\rangle_{s'_{j+1}} - \langle Y\rangle_{s'_j}\right)} \\
&= \left(\int_0^{\tau_k} f^2(s) \,\mathrm{d}\langle X\rangle_s\right)^{1/2} \left(\int_0^{\tau_k} g^2(s) \,\mathrm{d}\langle Y\rangle_s\right)^{1/2}.
\end{aligned}$$

This shows (5.3) and (5.4) for $f, g \in \mathscr{E}$ and the rest follows from density of $\mathscr{E}$ in $\mathrm{L}^2_{\mathrm{loc}}(\langle X\rangle)$ and $\mathrm{L}^2_{\mathrm{loc}}(\langle Y\rangle)$, cf. Theorem 3.13, using the following metric:

$$\sum_{k=0}^{\infty} 2^{-k} \mathbb{E}\left[1 \wedge \int_0^{\tau_k} f \,\mathrm{d}\langle X\rangle_s\right],$$

where we might take a subsequence to achieve density a.s. ■

We can characterise the stochastic integral very elegantly «geometrically».

Theorem 5.4. *Let $X \in \mathscr{M}^2$ and $f \in \mathrm{L}^2(\langle X\rangle) \cap \mathscr{B}(\mathscr{P})$. Then*

$$\forall Y \in \mathscr{M}^2: \qquad \left\langle \int_0^{\bullet} f \,\mathrm{d}X, Y \right\rangle_t = \int_0^t f(s) \,\mathrm{d}\langle X, Y\rangle_s. \tag{5.7}$$

Conversely, if for some $I \in \mathscr{M}^2$ it holds

$$\forall Y \in \mathscr{M}^2: \qquad \langle I, Y\rangle_t = \int_0^t f(s) \,\mathrm{d}\langle X, Y\rangle_s, \tag{5.8}$$

then we already have

$$I_t - I_0 = \int_0^t f(s) \,\mathrm{d}X_s.$$

Proof. Let $f \in \mathscr{E}$ given by the representation

$$f = \sum_{\text{finite}} \varphi_j \mathbb{1}_{(s_j, s_{j+1}]}.$$

WLOG let $s_n = s < t = s_N$ (it not refine the partition). We set $\mathsf{M} := f \bullet \mathsf{X}$. Then

$$\begin{aligned}
\mathbb{E}\big((M_t - M_s)(Y_t - Y_s) \;\big|\; \mathscr{F}_s\big) &\overset{\text{tower}}{\underset{(*)}{=}} \sum_{j=n}^{N-1} \mathbb{E}\Big(\mathbb{E}\Big((M_{s_{j+1}} - M_{s_j})(Y_{s_{j+1}} - Y_{s_j}) \;\Big|\; \mathscr{F}_{s_j}\Big) \;\Big|\; \mathscr{F}_s\Big) \\
&\overset{\text{pull}}{\underset{\text{out}}{=}} \sum_{j=n}^{N-1} \mathbb{E}\Big(\varphi_j \mathbb{E}\Big((X_{s_{j+1}} - X_{s_j})(Y_{s_{j+1}} - Y_{s_j}) \;\Big|\; \mathscr{F}_{s_j}\Big) \;\Big|\; \mathscr{F}_s\Big) \\
&\overset{(5.2)}{=} \sum_{j=n}^{N-1} \mathbb{E}\Big(\varphi_j \mathbb{E}\Big(\langle X, Y\rangle_{s_{j+1}} - \langle X, Y\rangle_{s_j} \;\Big|\; \mathscr{F}_{s_j}\Big) \;\Big|\; \mathscr{F}_s\Big) \\
&= \mathbb{E}\left(\sum_{j=n}^{N-1} \varphi_j \left(\langle X, Y\rangle_{s_{j+1}} - \langle X, Y\rangle_{s_j}\right) \;\middle|\; \mathscr{F}_s\right) \\
&= \mathbb{E}\left(\int_s^t f(u)\, \mathrm{d}\langle X, Y\rangle_u \;\middle|\; \mathscr{F}_s\right).
\end{aligned}$$

The equality marked by $(*)$ is not easy, but meanwhile a standard argument for us: Since $s_n = s < t = s_N$, we can use the following telescope sums:

$$M_t - M_s = \sum_{j=n}^{N-1} \left(M_{s_{j+1}} - M_{s_j}\right) \quad \text{and} \quad Y_t - Y_s = \sum_{j=n}^{N-1} \left(Y_{s_{j+1}} - Y_{s_j}\right).$$

Hence, we find for the LHS

$$\mathbb{E}\big((M_t - M_s)(Y_t - Y_s) \;\big|\; \mathscr{F}_s\big) = \sum_{j=n}^{N-1} \sum_{k=n}^{N-1} \mathbb{E}\Big(\Big(M_{s_{j+1}} - M_{s_j}\Big)\Big(Y_{s_{k+1}} - Y_{s_k}\Big) \;\Big|\; \mathscr{F}_s\Big).$$

Now we show that the mixed sums vanish except for the cases $j \neq k$. WLOG let $j < k$. Then $s_j < s_{j+1} \leqslant s_k < s_{k+1}$ and we find, by the tower property,

$$\begin{aligned}
\mathbb{E}\Big(\Big(M_{s_{j+1}} - M_{s_j}\Big)\Big(Y_{s_{k+1}} - Y_{s_k}\Big) \;\Big|\; \mathscr{F}_s\Big) &= \mathbb{E}\Big(\mathbb{E}\Big(\Big(M_{s_{j+1}} - M_{s_j}\Big)\Big(Y_{s_{k+1}} - Y_{s_k}\Big) \;\Big|\; \mathscr{F}_{s_k}\Big) \;\Big|\; \mathscr{F}_s\Big) \\
&\overset{\text{pull}}{\underset{\text{out}}{=}} \Big(M_{s_{j+1}} - M_{s_j}\Big) \mathbb{E}\Big(\mathbb{E}\Big(Y_{s_{k+1}} - Y_{s_k} \;\Big|\; \mathscr{F}_{s_k}\Big) \;\Big|\; \mathscr{F}_s\Big) \\
&\overset{\text{MG}}{=} \Big(M_{s_{j+1}} - M_{s_j}\Big) \cdot 0 = 0.
\end{aligned}$$

Hence, $(*)$ now follows by summation of the «pure» members, where $j = k$.

Thus, for all $f \in \mathscr{E}$,

$$\mathbb{E}\left(\left(\left(\int_0^t f(u)\, \mathrm{d}X_u - \int_0^s f(u)\, \mathrm{d}X_u\right)(Y_t - Y_s) \;\middle|\; \mathscr{F}_s\right) = \mathbb{E}\left(\int_s^t f(u)\, \mathrm{d}\langle X, Y\rangle_u \;\middle|\; \mathscr{F}_s\right). \tag{5.9}$$

For every sequence such that

$$\mathscr{E} \ni f_n \xrightarrow[n\to\infty]{\mathsf{L}^2(\langle X\rangle)} f \in \mathsf{L}^2(\langle X\rangle) \cap \mathscr{B}(\mathscr{P})$$

it follows that

$$\int_0^t f_n(u)\,\mathrm{d}X_u \xrightarrow{\mathrm{L}^2(\mathbb{P})} \int_0^t f(u)\,\mathrm{d}X_u$$

by construction of the stochastic integral Definition 3.10, together with

$$\left|\int_s^t \big(f(u)-f_n(u)\big)\,\mathrm{d}\langle X,Y\rangle_u\right| \overset{(5.4)}{\leqslant} \left(\int_s^t \left|f(u)-f_n(u)\right|^2\,\mathrm{d}\langle X\rangle_u\right)^{1/2}\left(\int_s^t \mathrm{d}\langle Y\rangle_u\right)^{1/2} \xrightarrow{n\to\infty} 0,$$

on other words, (5.9) holds for all $f\in\mathrm{L}^2(\langle X\rangle)\cap\mathscr{B}(\mathscr{P})$.

The analog of Lemma 3.3 – or by polarising and a direct application of Lemma 3.3 – now shows that (5.9) is actually equivalent to (5.7).

Let us now assume (5.8) which means, by (5.7), also

$$\begin{aligned} &\langle f\bullet X - I, Y\rangle \equiv 0 \qquad \forall Y\in\mathscr{M}^2\\ \Longrightarrow\ &\langle f\bullet X - I, f\bullet X - I\rangle \equiv 0\\ \overset{(3.2)\,(c)}{\Longrightarrow}\ & f\bullet X - I \equiv \text{ constant random variable} \end{aligned}$$

■

Corollary 5.5. *Let* $\mathrm{X}\in\mathscr{M}^2$, $f\in\mathrm{L}^2(\langle X\rangle)\cap\mathscr{B}(\mathscr{P})$ *and* $g\in\mathrm{L}^2(\langle f\bullet X\rangle)\cap\mathscr{B}(\mathscr{P})$. *Then* $fg\in\mathrm{L}^2(\langle X\rangle)\cap\mathscr{B}(\mathscr{P})$ *and*

$$g\bullet(f\bullet \mathrm{X}) = (gf)\bullet\mathrm{X}, \tag{5.10}$$

i.e.

$$\int_0^t g(s)\mathrm{d}\left(\int_0^\bullet f(u)\,\mathrm{d}X_u\right)_s = \int_0^t g(s)f(s)\,\mathrm{d}X_s. \tag{5.11}$$

Proof. First, notice that

$$\mathrm{X}\in\mathscr{M}^2, f\in\mathrm{L}^2(\langle X\rangle)\cap\mathscr{B}(\mathscr{P}) \quad\Longrightarrow\quad f\bullet\mathrm{X}\in\mathscr{M}^2_0.$$

By Theorem 5.4,

$$\langle f\bullet X\rangle_t = \int_0^t f^2(u)\,\mathrm{d}\langle X\rangle_u$$

and the rules for the Riemann-Stieltjes integral

$$\int_0^t g^2(u)\,\mathrm{d}\langle f\bullet X\rangle_u = \int_0^t g^2(u)f^2(u)\,\mathrm{d}\langle X\rangle_u$$

and thus $fg\in\mathrm{L}^2(\langle X\rangle)$. Again by Theorem 5.4 we now find, for all $\mathrm{Y}\in\mathscr{M}^2$,

$$\begin{aligned}\left\langle \int_0^\bullet g\mathrm{d}(f\bullet X), Y\right\rangle_t &\overset{(5.7)}{=} \int_0^t g(s)\,\mathrm{d}\langle f\bullet X, Y\rangle_s\\ &= \int_0^t g(s)f(s)\,\mathrm{d}\langle X,Y\rangle_s\\ &= \left\langle\int_0^\bullet gf\,\mathrm{d}X, Y\right\rangle_t,\end{aligned}$$

i.e. by Theorem 5.4 we already have $g\bullet f\bullet\mathrm{X} = (gf)\bullet\mathrm{X}$. ■

Recall that $(\mathcal{M}_0^2, \|\cdot\|)$ with $\|X\|^2_{\mathcal{M}^2} := \mathbb{E}X^2_\infty$ is a Hilbert space. Conversely, also

$$\mathbb{E}\langle X\rangle_t = \mathbb{E}X_t^2 \qquad \text{cf. Remark 3.2 (c)}$$

and thus

$$\mathbb{E}\langle X\rangle_\infty \overset{\text{BL}}{=} \sup_t \mathbb{E}\langle X\rangle_t = \sup_t \mathbb{E}X_t^2 \asymp \|X\|^2_{\mathcal{M}^2},$$

i.e. $(\mathcal{M}_0^2, \mathbb{E}\langle\cdot\rangle_\infty)$ is a Hilbert space with scalar product $(X,Y) \mapsto \mathbb{E}\langle X,Y\rangle_\infty$. Therefore, we can talk about the *geometry of the Hilbert space*!

Definition 5.6. A subset $\mathcal{N}$ of the Hilbert space $\mathcal{M}_0^2$ (with scalar product $\mathbb{E}\langle\cdot,\cdot\rangle_\infty$) is called **stable subspace** if

(1) $\mathcal{N}$ is a closed subspace;

(2) $\forall X \in \mathcal{N}$, $f \in \mathrm{L}^2(\langle X\rangle) \cap \mathcal{B}(\mathcal{P}) \implies f \bullet X \in \mathcal{N}$.

Example 5.7. Let $X \in \mathcal{M}^2$. Then

$$\mathcal{L}(X) := \left\{ f \bullet X : f \in \mathrm{L}^2(\langle X\rangle) \cap \mathcal{B}(\mathcal{P}) \right\}$$

is a stable subspace of $\mathcal{M}_0^2$.

Subspace: Clear.

Closed: Let $f_n \in \mathrm{L}^2(\langle X\rangle)$ mit $f_n \bullet X \xrightarrow[n\to\infty]{\mathcal{M}_0^2} M \in \mathcal{M}_0^2$. Then by Itô's isometry

$$\mathbb{E}\int_0^\infty \left(f_n(u) - f_m(u)\right)^2 \,\mathrm{d}\langle X\rangle_u \overset{(3.11)\,(f)}{=} \sup_t \mathbb{E}\left|\int_0^t \left(f_n(u) - f_m(u)\right)\,\mathrm{d}X_s\right|^2$$
$$\xrightarrow{n,m\to\infty} 0.$$
$$\implies \quad (f_n)_n \text{ is Cauchy in } \mathrm{L}^2(\langle X\rangle)$$
$$[\text{a.s. convergent subsequence gives } \mathcal{P}\text{-mb. limit}]$$
$$\implies \quad \exists f \in \mathrm{L}^2(\langle X\rangle) \cap \mathcal{B}(\mathcal{P}) : \quad f_n \xrightarrow{\mathrm{L}^2(\langle X\rangle)} f$$
$$\overset{(3.11)\,(f)}{\implies} \quad M = f \bullet X \in \mathcal{L}(X).$$

Stability: Let $f \bullet X \in \mathcal{L}(X)$ and $g \in \mathrm{L}^2(\langle f \bullet X\rangle) \cap \mathcal{B}(\mathcal{P})$. Then by Corollary 5.5 also $g \bullet f \bullet X = (fg) \bullet X$ with $fg \in \mathrm{L}^2(\langle X\rangle) \cap \mathcal{B}(\mathcal{P})$.

Definition 5.8. $X, Y \in \mathcal{M}_0^2$ are **orthogonal**, denoted $X \perp Y$, if $\langle X,Y\rangle_t \equiv 0$.

Equivalent $\quad X \perp Y \iff XY \in \mathcal{M}$.

Caution «$\perp$» is not in the sense of the scalar product on $\mathcal{M}_0^2$.

$$X \perp Y \implies \mathbb{E}\langle X,Y\rangle_\infty = 0 \qquad \text{but} \qquad \text{«}\Longleftarrow\text{» false i.g.!}$$

Example 5.9. Let $\mathscr{N} \subset \mathscr{M}_0^2$ a stable subspace. Then the **orthogonal complement**

$$\mathscr{N}^\perp := \left\{ \mathrm{Y} \in \mathscr{M}_0^2 : \langle X, Y \rangle_t \equiv 0 \quad \forall \mathrm{X} \in \mathscr{N} \right\}$$

is a stable subspace.

Subspace: Clear.

Closed: Exercise! – Tip: CSI for $\langle \cdot, \cdot \rangle$.

Stability: Let $\mathrm{Y} \in \mathscr{N}^\perp$ and $g \in \mathrm{L}^2(\langle Y \rangle) \cap \mathscr{B}(\tilde{}mP)$. Then

$$\langle g \bullet Y, X \rangle \overset{(5.7)}{=} \int_0^\bullet g \,\mathrm{d}\langle X, Y \rangle \overset{\langle X,Y \rangle = 0}{=} 0 \qquad \forall X \in \mathscr{N}.$$

Hence $g \bullet \mathrm{Y} \in \mathscr{N}^\perp$.

Our next goal is to show that, for fixed $\mathrm{X} \in \mathscr{M}_0^2$, every $\mathrm{Y} \in \mathscr{M}_0^2$ can be uniquely represented as the sums of $\mathrm{Y}^1 \in \mathscr{L}(\mathrm{X})$ and $\mathrm{Y}^2 \in \mathscr{L}(\mathrm{X})^\perp$. Therefore, we start with a

Lemma 5.10. *Let* $\mathrm{X}, \mathrm{Y} \in \mathscr{M}_0^2$*. Then there is a uniquely determined* $f \in \mathrm{L}^2(\langle X \rangle) \cap \mathscr{B}(\mathscr{P})$ *with*

$$\langle X, Y \rangle = \int_0^\bullet f \,\mathrm{d}\langle X \rangle .$$

Proof. Let $A \in \mathscr{P}$. Then

$$\int \mathbb{1}_A \,\mathrm{d}\langle X \rangle = 0 \quad \overset{5.3}{\Longrightarrow} \quad \int \mathbb{1}_A \,\mathrm{d}\,|\langle X, Y \rangle|_s = 0.$$

By Randon-Nikodým's theorem, we can choose an $f_1 \in \mathscr{P}$ such that

$$f_1(t, \omega) = \frac{\mathrm{d}\,|\langle X(\omega), Y(\omega) \rangle|_t}{\mathrm{d}\langle X(\omega) \rangle_t}.$$

Set

$$f_1^c(s) := f_1(s) \mathbb{1}_{\{|f| \leqslant c\}}.$$

Then it holds $(f_1^c)^2 = f_1 \cdot f_1^c$ and thus

$$\int_0^t (f_1^c(s))^2 \,\mathrm{d}\langle X \rangle_s = \int_0^t f_1^c(s) \,\mathrm{d}\,|\langle X, Y \rangle|_s \leqslant \sqrt{\int_0^t \left| f_1^c(s) \right|^2 \,\mathrm{d}\langle X \rangle_s} \sqrt{\langle Y \rangle_t},$$

and thus

$$\int_0^t \left| f_1^c(s) \right|^2 \,\mathrm{d}\langle X \rangle_s \leqslant \langle Y \rangle_t .$$

By Beppo Levi and $c \to \infty$, it follows that $f_1 \in \mathrm{L}^2(\langle X \rangle)$. We have shown:

$$\exists f_1 \in \mathrm{L}^2(\langle X \rangle) \cap \mathscr{B}(\mathscr{P}) : \quad V_t := \mathrm{var}(\langle \mathrm{X}, \mathrm{Y} \rangle ; [0, t]) = \int_0^t f_1(s) \,\mathrm{d}\langle X \rangle_s .$$

Finally, we consider $\langle X, Y \rangle_t = V_t - A_t$, where $V_t = \mathrm{var}(\langle \mathrm{X}, \mathrm{Y} \rangle ; [0, t])$, i.e. $\mathrm{d}V_t = \mathrm{d}\,|\langle X, Y \rangle|_t$. Then

$$A_t = V_t - \langle X, Y \rangle_t \leqslant V_t + |\langle X, Y \rangle|_t$$

and also absolutely continuous. The claim follows now by a similar argument as above, using $f = f_1 - f_2$. ■

Proposition 5.11. *Let* $\mathsf{X}, \mathsf{Y} \in \mathscr{M}_0^2$*. Then there is a unique decomposition*

$$\mathsf{Y} = \mathsf{Y}^1 + \mathsf{Y}^2, \qquad \mathsf{Y}^1 \in \mathscr{L}(\mathsf{X}), \quad \mathsf{Y}^2 \in \mathscr{L}(\mathsf{X})^\perp.$$

Proof. By Lemma 5.10, we know that

$$\langle Y, X\rangle_t = \int_0^t f(s)\,\mathrm{d}\langle X\rangle_s \qquad \text{for some} \quad f \in \mathrm{L}^2(\langle X\rangle) \cap \mathscr{B}(\mathscr{P}).$$

Set

$$\mathsf{Y}^1 := \int_0^\bullet f\,\mathrm{d}X, \qquad \mathsf{Y}^2 := \mathsf{Y} - \mathsf{Y}^1.$$

Obviously, $\mathsf{Y}^1 \in \mathscr{L}(\mathsf{X})$. By

$$\left\langle Y^1, X\right\rangle \overset{5.4}{=} \int f\,\mathrm{d}\langle X, Y\rangle \overset{\text{def}}{=} \langle Y, X\rangle$$

we get $\left\langle Y^2, X\right\rangle = 0 \quad \Longrightarrow \quad \mathsf{Y}^2 \in \mathscr{L}(\mathsf{X})^\perp$, using examples 5.7 and 5.9.

By Uniqueness,

$$\mathsf{Y}^1 + \mathsf{Y}^2 = \tilde{\mathsf{Y}}^1 + \tilde{\mathsf{Y}}^2 \quad \Longrightarrow \quad \underbrace{\mathsf{Y}^1 - \tilde{\mathsf{Y}}^1}_{\in \mathscr{L}(X)} = \underbrace{\tilde{\mathsf{Y}}^2 - \mathsf{Y}^2}_{\in \mathscr{L}(X)^\perp} = 0.$$ ■

Definition 5.12. The element $\mathsf{Y}^1 \in \mathscr{L}(X)$, defined in Proposition 5.11, is called **orthogonal projection** of $\mathsf{Y} \in \mathscr{M}_0^2$ on $\mathscr{L}(X)$. We write $\mathsf{Y}^1 := P_{\mathscr{L}(X)}\mathsf{Y}$.

Remark 5.13. Orthogonality in the sense of Definition 5.12 is **NOT** orthogonality in the sense of Hilbert space geometry, because we do not use the scalar product of the Hilbert space. However, there are cases where both notions concide.

Proposition 5.14. *Let* $\mathscr{N} \subset \mathscr{M}_0^2$ *be a stable subspace. Then*

$$\mathscr{N}^\perp = \left\{\mathsf{Y} \in \mathscr{M}_0^2 : \mathbb{E}\langle X, Y\rangle = 0 \quad \forall \mathsf{X} \in \mathscr{N}\right\}. \tag{5.12}$$

Proof. Let us denote the RHS of (5.12) by $\tilde{\mathscr{N}}$. By

$$\begin{aligned}\langle X, Y\rangle_t = 0 \quad &\Longrightarrow \quad \mathbb{E}\langle X, Y\rangle_t = 0 \\ &\Longrightarrow \quad \mathscr{N}^\perp \subset \tilde{\mathscr{N}}.\end{aligned}$$

Not, let $\mathsf{Y} \in \tilde{\mathscr{N}}$. Then

$$\mathbb{E}\langle X, Y\rangle_t = 0 \quad \overset{5.2}{\Longrightarrow} \quad \mathbb{E}\big[X_t Y_t\big] = 0 \qquad \forall X \in \mathscr{N}.$$

Since $\mathscr{N}$ is stable, we have for all bounded ST τ

$$X_t^\tau = \int_0^{t\wedge\tau} \mathrm{d}X_s = \int_0^t \mathbb{1}_{[0,\tau]}(s)\,\mathrm{d}X_s,$$

and as $\mathbb{1}_{[0,\tau]} \in \mathrm{L}^2(\langle X\rangle) \cap \mathscr{B}(\mathscr{P})$, also $X^\tau \in \mathscr{N}$. Thus

$$\begin{aligned} \mathbb{E}\,\langle X^\tau, Y\rangle_t = 0 \quad &\Longrightarrow \quad \mathbb{E}\,\langle X, Y\rangle_{t\wedge\tau} = 0 \\ &\Longrightarrow \quad \mathbb{E}\big[X_{t\wedge\tau} Y_{t\wedge\tau}\big] = 0 \\ &\overset{\tau}{\underset{\text{bdd}}{\Longrightarrow}} \quad \mathbb{E}\big[X_\tau Y_\tau\big] = 0 \\ &\Longrightarrow \quad (X_t Y_t)_{t\geqslant 0} \quad \text{is MG, cf. Theorem 2.7} \\ &\Longrightarrow \quad \langle X, Y\rangle_t = 0, \end{aligned}$$

where we used Lemma 5.2 and the equivalent of Lemma 3.3. Hence, it follows $Y \in \mathscr{N}^\perp$ and $\tilde{\mathscr{N}} \subset \mathscr{N}^\perp$. ■

Corollary 5.15. *Let $\mathscr{N} \subset \mathscr{M}_0^2$ a stable subspace. Then the decompsition*

$$\mathscr{M}_0 = \mathscr{N} \oplus^\perp \mathscr{N}^\perp$$

is unique (and is the «normal» orthogonal decomposition).

Example 5.16. Let

$$\mathscr{M}_{0,c}^2 = \left\{ X \in \mathscr{M}_0^2 : t \mapsto X_t \quad \text{a.s. continuous} \right\}.$$

Clearly, $\mathscr{M}_{0,c}^2$ is a subspace of $\mathscr{M}_0^2$ and, since

$$\mathbb{E}\,\langle X\rangle_\infty \asymp \mathbb{E} \sup_{t\geqslant 0} X_t^2,$$

it is also closed.

For $f \in \mathscr{E}$ with $f(s,\omega) = \sum_j \varphi_j(\omega)\mathbb{1}_{(t_j,t_{j+1}]}(s,\omega)$, we have

$$f \bullet X_t = \sum_j \varphi_j \left(X_{t_{j+1}\wedge t} - X_{t_j \wedge t} \right) \in \mathscr{M}_{0,c}^2.$$

(Mind: $\varphi_j = f(t_{j-1})$). Conversely, $\mathscr{E}$ is dense in $\mathrm{L}^2(\langle X\rangle) \cap \mathscr{B}(\mathscr{P})$ and we find for every sequence $f_n \xrightarrow{\mathrm{L}^2(\langle X\rangle)} f$, where $f_n \in \mathscr{E}$, thus

$$\begin{aligned} \mathbb{E}\left[\sup_{t\geqslant 0} \left| f_n \bullet X_t - f \bullet X_t \right|^2 \right] &\overset{\text{Doob}}{\leqslant} 4 \sup_{t\geqslant 0} \mathbb{E} \left| f_n \bullet X_t - f \bullet X_t \right|^2 \\ &\overset{\text{Itô}}{\leqslant} 4 \sup_{t\geqslant 0} \mathbb{E} \int_0^t \left| f_n(u) - f(u) \right|^2 \,\mathrm{d}\,\langle X\rangle_u \\ &\overset{\text{BL}}{\leqslant} 4\mathbb{E} \int_0^\infty \left| f_n(u) - f(u) \right|^2 \,\mathrm{d}\,\langle X\rangle_u, \end{aligned}$$

i.e. $f \bullet X$ inherits the a.s. continuous paths of the continuity of $f_n \bullet X$ (identify the $\mathscr{M}_0^2$-limit with a.s. convergent subsequences!).

Hence also $\mathscr{M}_{0,c}^2$ is a stable subspace of $\mathscr{M}_0^2$.

Definition 5.17. $\mathscr{M}_{0,d}^2 := (\mathscr{M}_{0,c}^2)^\perp$ is the set of all purely discontinuous L^2-MG.

Mind $\mathscr{M}_0^2 = \mathscr{M}_{0,c}^2 \oplus^\perp \mathscr{M}_{0,d}^2$. The decomposition is unique.

Chapter 6

The «Square Bracket»

In §§ 3 and 5 we have been acquainted with the angle bracket = *previsible/predictable* quadratic variation $\langle X\rangle$, resp. $\langle X,Y\rangle$. We now want to extend the notion to a *non-previsible* version.

Since a SMG $(X_t)_{t\geqslant 0}$ is càdlàg, $X_- := (X_{s-})_{s\geqslant 0}$ is defined and left continuous, hence $\mathscr{P}$-measurable. Thus also

$$\int_0^t X_{s-}\,\mathrm{d}X_s =: X_- \bullet X_t \tag{6.1}$$

is well-defined, cf. Theorem 4.8.

Definition 6.1. Let $(X_t)_{t\geqslant 0}$ be a SMG. Then we call

$$[X]_t := X_t^2 - X_0^2 - 2\int_0^t X_{s-}\,\mathrm{d}X_s \tag{6.2}$$

the **quadratic variation** or **square bracket** of the SMG $(X_t)_{t\geqslant 0}$. The **quadratic covariation** is defined by

$$[X,Y]_t := \frac{1}{4}\left([X+Y]_t - [X-Y]_t\right). \tag{6.3}$$

Remark 6.2. (a) If $X \in \mathscr{M}_0^2$, then (6.2) says that

$$X_t^2 - [X]_t = 2\int_0^t X_{s-}\,\mathrm{d}X_s$$

and thus a MG.

Caution: $[X]_t$ does not have to be $\mathscr{P}$-measurable! Else we would have $[X] = \langle X\rangle$.

(b) $[X]_t$ is positive, increasing and adapted (cf. Lemma 6.3 below).

(c) For $X \in \mathscr{M}^2$, we have

$$\mathbb{E}\,[X]_\infty^2 = \mathbb{E}X_\infty^2 < \infty$$

and thus $([X]_t)_{t\geqslant 0}$ is a ui sub-MG (trivial: increasing and $[X]_t \leqslant [X]_\infty \in L^1(\mathbb{P})$). By the Doob-Meyer decomposition 1.13 thus

$$[X]_t = M_t + A_t,$$

where $A \in \mathscr{B}(\mathscr{P})$ is increasing and $M \in \mathscr{M}$. But also

$$\begin{aligned}[X]_t &= X_t^2 - 2\int_0^t X_{s-}\,\mathrm{d}X_s \\ &= \underbrace{\left(X_t^2 - \langle X\rangle_t\right) - 2\int_0^t X_{s-}\,\mathrm{d}X_s}_{\in\mathscr{M}} + \underbrace{\langle X\rangle_t}_{\in\mathscr{P}}\end{aligned}$$

and hence $A = \langle X \rangle$, so

$$[X] - \langle X \rangle \qquad \text{is a martingale!}$$

For a partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n < \infty\}$, we set

$$[X,Y]_t^\pi = \sum_{j=0}^{n-1} \left(X_{t_{j+1}\wedge t} - X_{t_j \wedge t} \right) \left(Y_{t_{j+1}\wedge t} - Y_{t_j\wedge t} \right) \tag{6.4}$$

(cf. notation in Theorem 4.8!).

Lemma 6.3. *Let π_k a sequence of partitions with $|\pi_k| \xrightarrow{k\to\infty} 0$. Then for SMG $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$*

$$[X,Y]_t = \text{ucp} - \lim_{k\to\infty} [X,Y]_t^{\pi_k}.$$

Proof. WLOG X = Y, else: use polarisation. By Theorem 4.8 and Corollary 4.10 we know that

$$X_-^{\pi_k} \bullet X \xrightarrow[k\to\infty]{\text{ucp}} X_- \bullet X.$$

By construction for $\pi = \pi_k = \{0 = t_0 < t_1 < \ldots < t_n\}$ (note we suppress the k-dependence of n and t_j!)

$$\begin{aligned}
[X]_t^\pi &= \sum_{j=0}^{n-1} (X_{t_{j+1}\wedge t} - X_{t_j\wedge t})^2 \\
&= \sum_{j=0}^{n-1} X_{t_{j+1}\wedge t}(X_{t_{j+1}\wedge t} - X_{t_j\wedge t}) - \sum_{j=0}^{n-1} X_{t_j\wedge t}(X_{t_{j+1}\wedge t} - X_{t_j\wedge t}) \\
&= \sum_{j=0}^{n-1} (X_{t_{j+1}\wedge t} + X_{t_j\wedge t})(X_{t_{j+1}\wedge t} - X_{t_j\wedge t}) - 2\sum_{j=0}^{n-1} X_{t_j\wedge t}(X_{t_{j+1}\wedge t} - X_{t_j\wedge t}) \\
&= \sum_{j=0}^{n-1} (X^2_{t_{j+1}\wedge t} - X^2_{t_j\wedge t}) - 2X_-^\pi \bullet X_t \\
&= X_t^2 - X_0^2 - 2X_-^\pi \bullet X_t \\
&\xrightarrow[|\pi|\to 0,\, 4.10]{\text{ucp}} X_t^2 - X_0^2 - 2X_- \bullet X_t
\end{aligned}$$

and the claim follows. ■

Theorem 6.4. *Let $X \in \mathcal{M}_0^2$.*

(a) $X \in \mathcal{M}^2_{0,c} \implies [X] = \langle X \rangle$;

(b) $X \in \mathcal{M}^2_{0,d} \implies [X] = \sum_{s\leqslant t} (\Delta X_s)^2$, *where* $\Delta X_s = X_s - X_{s-}$;

(c) For $X = X^c + X^d \in \mathcal{M}^2_{0,c} \oplus \mathcal{M}^2_{0,d}$, *we get*

$$[X]_t = \langle X^c \rangle_t + [X^d]_t = \langle X^c \rangle_t + \sum_{s\leqslant t} (\Delta X_s)^2.$$

Proof. (a) Let $(X_t)_{t\geqslant 0}$ be continuous, then

$$\begin{aligned}
&\Longrightarrow && \int_0^t X_{s-}\,\mathrm{d}X_s \quad \text{continuous, cf. Example 5.16} \\
&\Longrightarrow && [X]_t = X_t^2 - X_0^2 - 2X_- \bullet X_t \quad \text{continuous and thus } mP\text{-mb} \\
&\Longrightarrow && [X]_t - X_t^2 \quad \text{is MG and} \quad [X] = \langle X\rangle \\
& && \big[\text{uniqueness of } [X]\big]
\end{aligned}$$

(b) follows from (c)

(c) Let $X_t = X_t^c + X_t^d$. Then

$$[X]_t = \left[X^c\right]_t + 2\left[X^c, X^d\right]_t + \left[X^d\right]_t$$

und, since $\left\langle X^d, X^c\right\rangle_t = 0$ follows $\left[X^d, X^c\right]_t = 0$.

Indeed:

- $\left[X^c, X^d\right]_t$ is MG – by Remark 6.2 (c);
- $\Delta\left[X^c, X^d\right]_t = 0$ – by Lemma 6.5;
- $\left[X^c, X^d\right] \in \mathrm{BV}$ – as difference of increasing processes.

$\Longrightarrow \quad \left[X^c, X^d\right] = 0$, cf. Corollary 4.7.

Hence, we know that

$$\begin{aligned}
[X]_t &= \left[X^c\right]_t + \left[X^d\right]_t && \text{unique, above} \\
&= [X]_t^c + [X]_t^d && \text{unique, by BV} \\
&= [X]_t^c + \sum_{s\leqslant t} \Delta [X]_s && \text{saltus function for BV} \\
&= [X]_t^c + \sum_{s\leqslant t} (\Delta X_s)^2 && \text{by Lemma 6.5.}
\end{aligned}$$

■

Lemma 6.5. $\Delta [X]_t = (\Delta X_s)^2$ *and* $\Delta [X, Y]_t = \Delta X_t \Delta Y_t$.

Proof. The second formula follows from the first by polarisation. For the first formula, consider

$$\begin{aligned}
\Delta [X]_t &= \Delta\left(X^2 - 2X_- \bullet X\right)_t \\
&= \Delta X_t^2 - 2X_{t-}\Delta X_t && \text{by Proposition 3.11 (g)} \\
&= X_t^2 - X_{t-}^2 - 2X_{t-}\Delta X_t \\
&= (X_t + X_{t-})\Delta X_t - 2X_{t-}\Delta X_t \\
&= (\Delta X_t)^2.
\end{aligned}$$

■

Remark 6.6. (a) $X, Y \in \mathscr{M}^2_{0,c}$. Then

$$X \perp Y \quad \Longleftrightarrow \quad \langle X, Y\rangle_t \equiv 0 \quad \overset{6.4}{\Longleftrightarrow} \quad [X, Y]_t \equiv 0.$$

(b) $X, Y \in \mathscr{M}^2_{0,d}$. Then

$$[X,Y]_t \equiv 0 \iff \Delta X_t \Delta Y_t \equiv 0 \text{ – hence no } \textbf{mutual} \text{ jumps}$$
$$\implies X \perp Y.$$

Caution: $X \perp Y \quad \not\Longrightarrow \quad \Delta X_t \Delta Y_t = 0.$

Finally, we show a necessary and sufficient condition that a local MG is a (proper) MG. We also improve the well-known criterion in Theorem 2.14.

Theorem 6.7. *Let $X \in \mathscr{M}_{\text{loc}}$. Then*

$$X \in \mathscr{M}, \quad \mathbb{E}X_t^2 < \infty \quad \forall t \geqslant 0 \qquad \iff \qquad \mathbb{E}\,[X,X]_t < \infty \quad \forall t \geqslant 0.$$

If $\mathbb{E}\,[X,X]_t < \infty$, then $\mathbb{E}X_t^2 = \mathbb{E}\,[X,X]_t$.

Proof. $\implies$ Let $X \in \mathscr{M}$ with $\mathbb{E}X_t^2 < \infty$ for all $t \geqslant 0$. Clearly, then $X \in \mathscr{M}^2_{\text{loc}}$ and

$$Y_t := X_t^2 - [X,X]_t = 2\int_0^t X_{s-}\,\mathrm{d}X_s$$

is again in $\mathscr{M}^2_{\text{loc}}$ by Definition 4.2. Let τ_n be a reducing sequence such that $Y^{\tau_n} \in \mathscr{M}^2$. Then $\mathbb{E}Y_t^{\tau_n} = \mathbb{E}Y_0 = 0$. Thus

$$\mathbb{E}X^2_{t\wedge\tau_n} = \mathbb{E}\,[X,X]_{t\wedge\tau_n}$$

and by Doob's inequality then

$$\mathbb{E}\sup_{s\leqslant t} X_s^2 \leqslant 4\mathbb{E}X_t^2 < \infty.$$

By dominated convergence hence

$$\mathbb{E}X_t^2 = \lim_{n\to\infty} \mathbb{E}X^2_{t\wedge\tau_n} = \lim_{n\to\infty} \mathbb{E}\,[X,X]_{t\wedge\tau_n} \overset{\text{BL}}{=} \mathbb{E}\,[X,X]_t < \infty.$$

$\impliedby$ Now, let $\mathbb{E}\,[X,X]_t < \infty$ for all $t \geqslant 0$. Then

$$\tau_n := \inf\left\{t > 0 : \left|M_t\right| > n\right\} \wedge n$$

are ST and $\tau_n \xrightarrow{\text{a.s.}} \infty$. Further,

$$\sup_s \left|X_s^{\tau_n}\right| \leqslant \sup_{s<\tau_n} \left|X_s\right| + \left|\Delta X_{\tau_n}\right| \leqslant n + \left|\Delta X_{\tau_n}\right| \leqslant n + \sqrt{[X,X]_n} \in \mathrm{L}^2(\mathbb{P}).$$

Thus X^{τ_n} is a ui MG for every n. Moreover,

$$\mathbb{E}\left|X_t^{\tau_n}\right|^2 \leqslant \mathbb{E}\sup_s \left|X_s^{\tau_n}\right| < \infty, \qquad t \geqslant 0.$$

Using the first part of this theorem, this means

$$\mathbb{E}\left|X_t^{\tau_n}\right|^2 = \mathbb{E}\left[X^{\tau_n}, X^{\tau_n}\right]_t$$

and by Doob's inequality

$$\mathbb{E}\sup_{s\leqslant t\wedge\tau_n} \left|X_s\right|^2 \leqslant 4\mathbb{E}\left|X_t^{\tau_n}\right|^2 = \mathbb{E}\left[X_t^{\tau_n}, X_t^{\tau_n}\right]_t = \mathbb{E}\,[X,X]_{t\wedge\tau_n} \leqslant \mathbb{E}\,[X,X]_t\,.$$

With Beppo Levi, finally

$$\mathbb{E}\sup_{s\leqslant t}\left|X_s\right|^2 = \lim_{n\to\infty}\mathbb{E}\sup_{s\leqslant t\wedge\tau_n}\left|X_s\right|^2 \leqslant \mathbb{E}\,[X,X]_t < \infty,$$

and the claim follows by Theorem 2.14.

■

Chapter 7

The Itô lemma (part 1)

… also known as *change-of-variables* formula or *Itô's formula*. The Itô lemma is the stochastic counterpart of the chain rule or fundamental theorem of calculus. Consequently, a main application will be solving SDEs (stochastic *d*ifferential *e*quations) and the specific calculation of stochastic integrals.

Theorem 7.1 (K. Itô). *Let $X_t = (X_t^1, \dots, X_t^d)$ a vector d SMG and $F \in C^2(\mathbb{R}^d, \mathbb{R})$. Then $F(X_t)$ is again a SMG and we have*

$$\begin{aligned} F(X_t) - F(X_0) = \sum_{j=1}^{d} \int_0^t \frac{\partial F}{\partial x_j}(X_{s-})\, \mathrm{d}X_s^j + \frac{1}{2} \sum_{j,k=1}^{d} \int_0^t \frac{\partial^2 F}{\partial x_j \partial x_k}(X_{s-})\, \mathrm{d}\left[X^j, X^k\right]_s^c \\ + \sum_{0<s\leqslant t} \left\{ F(X_s) - F(X_{s-}) - \sum_{j=1}^{d} \frac{\partial F}{\partial x_j}(X_{s-}) \Delta X_s^j \right\}. \end{aligned} \tag{7.1}$$

Here, $\left[X^j, X^k\right]_t^c$ denotes the continuous part of $\left[X^j, X^k\right]_t$, i.e. $\left\langle X^{j,c}, X^{k,c} \right\rangle_t$. Before proving (7.1) let us make two introductory remarks.

Remark 7.2. (a) Because $s \mapsto X_s$ is càdlàg, every path has (for fixed ω) only countable many jumps. Therefore, the sum appearing in (7.1) is also countable.

Even more holds true: By Taylor's formula, we know that

$$\begin{aligned} &\left| F(X_s) - F(X_{s-}) - \sum_{j=1}^{d} \frac{\partial F}{\partial x_j}(X_{s-}) \Delta X_s^j \right| \\ &\quad \leqslant \left| \frac{1}{2} \sum_{j,k=1}^{d} \int_0^s (1-\vartheta) \frac{\partial^2 F}{\partial x_j \partial x_k}(X_{s-} + \vartheta \Delta X_s) \mathrm{d}\vartheta\, \Delta X_s^j \Delta X_s^k \right| \\ &\quad \leqslant \frac{1}{2} K(\omega, t) \sum_{j,k=1}^{d} \left| \Delta X_s^j \Delta X_s^k \right| \\ &\quad \leqslant \frac{1}{4} K(\omega, t) \sum_{j,k=1}^{d} \left((\Delta X_s^j)^2 + (\Delta X_s^k)^2 \right), \end{aligned}$$

hence, by Theorem 6.4 (c), it follows

$$\left| \sum_{0<s\leqslant t} \left\{ \dots \right\} \right| \leqslant \frac{d}{2} K(\omega, t) \sum_{j=1}^{d} \left[X^j, X^j\right]_t < \infty.$$

(b) Since

$$\left[X^j, X^k\right]_t^d = \sum_{0<s\leqslant t} \Delta X_s^j \Delta X_s^k$$

and

$$\left[X^j, X^k\right]_t = \left[X^j, X^k\right]_t^c + \left[X^j, X^k\right]_t^d,$$

we can rewrite formula (7.1) in the following form:

$$F(X_t) - F(X_0) = \sum_{j=1}^{d} \int_0^t \frac{\partial F}{\partial x_j}(X_{s-})\, \mathrm{d}X_s^j + \frac{1}{2} \sum_{j,k=1}^{d} \int_0^t \frac{\partial^2 F}{\partial x_j \partial x_k}(X_{s-})\, \mathrm{d}\left[X^j, X^k\right]_s$$
$$+ \sum_{0<s\leqslant t} \Big\{ F(X_s) - F(X_{s-}) - \sum_{j=1}^{d} \frac{\partial F}{\partial x_j}(X_{s-}) \Delta X_s^j \qquad (7.2)$$
$$- \frac{1}{2} \sum_{j,k=1}^{d} \frac{\partial^2 F}{\partial x_j \partial x_k}(X_{s-}) \Delta X_s^j \Delta X_s^k \Big\}.$$

Proof (of Theorem 7.1). We show (7.2) for $d = 1$ (the case $d > 1$ just relies on heavier notation). First let us assume $F \in C_b^2$. Consequently all integrals on the RHS of (7.2) are well-defined. Further, let $\pi_k = \left\{0 = t_0^k < t_1^k < \ldots < t_{n(k)}^k = t\right\}$ s sequence of partitions of $[0, t]$ with $|\pi_k| \to 0$. We suppress the k-dependence from now on.

$$\begin{aligned} F(X_t) - F(X_0) &= \sum_{j=0}^{n-1} \left(F(X_{t_{j+1}}) - F(X_{t_j}) \right) \\ &= \sum_{j=0}^{n-1} F'(X_{t_j})(X_{t_{j+1}} - X_{t_j}) + \frac{1}{2} \sum_{j=0}^{n-1} F''(X_{t_j})(X_{t_{j+1}} - X_{t_j})^2 \\ &\quad + \sum_{j=1}^{n-1} R(X_{t_j}, X_{t_{j+1}}) \\ &\equiv I_1 + I_2 + I_3, \end{aligned}$$

where we used Taylor's formula with integral remainder:

$$F(y) - F(x) = F'(x)(y - x) + \frac{1}{2} F''(x)(y - x)^2 + R(x, y),$$

so

$$R(x, y) = \left(\int_0^1 (1 - \vartheta) F''(x + \vartheta(y - x))\mathrm{d}\vartheta \right) (y - x)^2 - \frac{1}{2} F''(x)(y - x)^2.$$

For I_1 holds by Corollary 4.10:

$$\text{ucp} - \lim_{k\to\infty} \sum_{j=0}^{n-1} F'(X_{t_j})(X_{t_{j+1}} - X_{t_j}) = \int_0^t F'(X_{s-})\, \mathrm{d}X_s.$$

For I_2 holds by Lemma 6.3 (with $Y = F''(X_-) \bullet X$):

$$\text{ucp} - \lim_{k\to\infty} \frac{1}{2} \sum_{j=0}^{n-1} F''(X_{t_j})(X_{t_{j+1}} - X_{t_j})^2 = \frac{1}{2} \int F''(X_{s-})\, \mathrm{d}\,[X]_s.$$

Finally to I_3: Set

$$J(\varepsilon) := \left\{ s \in [0,t] \,:\, \left|\Delta X_s\right| > \varepsilon \right\}.$$

Since $s \mapsto X_s$ is càdlàg, we have $\sharp J(\varepsilon) < \infty$ a.s. for every $\varepsilon > 0$. Thus

$$R(X_{t_j}, X_{t_{j+1}}) = \int_0^1 (1-\vartheta) F''(X_{t_j} + \vartheta(X_{t_{j+1}} - X_{t_j}))\mathrm{d}\vartheta \cdot (X_{t_{j+1}} - X_{t_j})^2 \\ - \frac{1}{2} F''(X_{t_{j+1}})(X_{t_{j+1}} - X_{t_j})^2.$$

Thus

$$\sum_{j : J(\varepsilon) \cap (t_j, t_{j+1}] \neq \emptyset} R(X_{t_j}, X_{t_{j+1}}) \\ \xrightarrow{k\to\infty} \sum_{s \in J(\varepsilon)} \int_0^1 (1-\vartheta) F''(X_{s-} + \vartheta \Delta X_s) \mathrm{d}\vartheta (\Delta X_s)^2 - \frac{1}{2} F''(X_{s-})(\Delta X_s)^2 \\ \overset{\text{Taylor}}{=} \sum_{s \in J(\varepsilon)} \left\{ F(X_s) - F(X_{s-}) - F'(X_{s-})\Delta X_s - \frac{1}{2} F''(X_{s-})(\Delta X_s)^2 \right\}.$$

Conversely, for all remaining indices:

$$\left| \sum_{j : J(\varepsilon) \cap (t_j, t_{j+1}] = \emptyset} R(X_{t_j}, X_{t_{j+1}}) \right| \\ \leqslant \sup_{j : J(\varepsilon) \cap (t_j, t_{j+1}] = \emptyset} \int_0^1 (1-\vartheta) \left| F''\left(X_{t_j} + \vartheta(X_{t_{j+1}} - X_{t_j})\right) - F''(X_{t_j}) \right| \mathrm{d}\vartheta \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})^2 \\ \leqslant \delta(\omega) \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})^2 \qquad \forall k \,:\, \left|\pi_k\right| < h,$$

where we used that $F'' \in \mathrm{C}_b$ and $s \mapsto X_s$ is càdlàg and so, in particular, $s \mapsto F(X_s)$ is uniformly continuous in $[0,t]$.

Therefore, by means of ucp-convergence:

$$\limsup_{k\to\infty} \left| \sum_{j : J(\varepsilon) \cap (t_j, t_{j+1}] = \emptyset} R(X_{t_j}, X_{t_{j+1}}) \right| \leqslant \delta(\omega)\ [X]_t \xrightarrow[h\to 0]{\varepsilon\to 0} 0.$$

Since also

$$\sum_{s \in J(\varepsilon)} \left\{ \cdots \right\} \xrightarrow[\varepsilon \to 0]{\text{a.s.}} \sum_{s \in (0,t]} \left\{ \cdots \right\},$$

we have shown formula (7.2) for $F \in \mathrm{C}_b^2$.

For the general case, let $F \in \mathrm{C}^2$. Then there is sequence $(F_m)_m \subset \mathrm{C}_b^2$ such that

$$F_m \xrightarrow{m\to\infty} F, \qquad F_m' \xrightarrow{m\to\infty} F', \qquad F_m'' \xrightarrow{m\to\infty} F''$$

locally uniformly.[1] Thus the assumption of Corollary 4.11 are satisfied and the first two terms in (7.1) converge to

$$\int_0^t F_m'(X_{s-})\,\mathrm{d}X_s \qquad \text{and} \qquad \frac{1}{2}\int F_m''(X_{s-})\,\mathrm{d}[X]_s$$

[1] Set e.g. $F_m := \chi_m F$, where $\chi_m \uparrow 1$ in $\mathrm{C}^2_{\mathrm{loc}}$.

in the ucp-sense.

The convergence of the third term

$$\sum_{0<s\leqslant t}\Big\{F_m(X_s) - F_m(X_{s-}) - F_m'(X_{s-})\Delta X_s^j\Big\}$$

follows by means of the estimate in Remark 7.2 (a), if we replace $F \rightsquigarrow F - F_m$ and use the locally uniform convergence for $m \to \infty$ for every ω. ■

An application: P. Lévy's Characterisation of Brownian Motion

In the hands of H. Kunita and S. Watanabe, the Itô formula became an extremely powerful tool. We follows their exposition and give a rather elegant proof for the following result by P. Lévy:

Theorem 7.3 (Lévy, 1948). *Let $X = (X_t)_{t\geqslant 0}$ be an $\mathcal{M}^{2,c}_{0,\mathrm{loc}}$-MG with quadratic variation $\langle X\rangle_t \equiv t$. Then X is already a Brownian motion.*

Mind Since X has continuous paths, we have $\langle X\rangle_t = [X]_t$.

Proof. We must show that, for all $s \leqslant t$,

$$X_t - X_s \perp\!\!\!\perp \mathscr{F}_s \qquad \text{and} \qquad X_t - X_s \sim \nu_{0,t-s}.$$

Therefore, it suffices to show that

$$\mathbb{E}\Big[e^{i\xi(X_t - X_s)}\mathbb{1}_F\Big] = e^{-\frac{1}{2}(t-s)\xi^2}\,\mathbb{P}(F)\,, \qquad \forall F \in \mathscr{F}_s. \tag{7.3}$$

Indeed: $F = \Omega$ results in

$$\mathbb{E}e^{i\xi(X_t - X_s)} = e^{-\frac{1}{2}(t-s)\xi^2},$$

i.e. $X_t - X_s$ is a Gaussian random variable, $\sim \nu_{0,t-s}$. In particular, (7.3) then becomes

$$\mathbb{E}\Big[e^{i\xi(X_t - X_s)}\mathbb{1}_F\Big] = \mathbb{E}\Big[e^{i\xi(X_t - X_s)}\Big]\,\mathbb{P}(F)\,, \qquad \forall F \in \mathscr{F}_s,$$

whereof it follows that $X_t - X_s \perp\!\!\!\perp F \in \mathscr{F}_s$.[2]

We now show (7.3). Obvisouly,

$$f(x) := e^{ix\xi}, \qquad f'(x) := i\xi e^{ix\xi}, \qquad f''(x) := -\xi^2 e^{ix\xi}.$$

[2] More precisely: Multiply both sides by the Fourier transform $\hat{f}(\xi)$ of an arbitrary function $f \in C_c^\infty$, integrate with respect to ξ and use Fubini's theorem to see

$$\mathbb{E}\big[f(X_t - X_s)\mathbb{1}_F\big] = \mathbb{E}f(X_t - X_s)\mathbb{E}\mathbb{1}_F.$$

We now approximate a closed interval I from above by a sequence $f_n \in C_c^\infty$. But using monotone convergence shows independence:

$$\begin{aligned}\mathbb{P}\big(\{X_t - X_s \in I\} \cap F\big) = \mathbb{E}\big[\mathbb{1}_I(X_t - X_s)\mathbb{1}_F\big] &= \mathbb{E}\big[\mathbb{1}_I(X_t - X_s)\big]\,\mathbb{E}\mathbb{1}_F \\ &= \mathbb{P}\big(X_t - X_s \in I\big)\,\mathbb{P}(F)\,.\end{aligned}$$

Applying Itô's formula (7.1) for the C^2 function $f(x) := e^{i\xi x}$, we find

$$e^{i\xi X_t} - e^{i\xi X_s} = i\xi \int_s^t e^{i\xi X_u}\,\mathrm{d}X_u - \frac{\xi^2}{2}\int_s^t e^{i\xi X_u}\mathrm{d}u. \tag{7.4}$$

Let $T > 0$ be fixed. Since $|e^{i\eta}| = 1$ the real and imaginary parts are in $L^2(\langle X\rangle^T)$ – mind: $\langle X\rangle_t = t$, i.e. $\langle X\rangle_t^T = t \wedge T$ – and we see that the stochastic integral appearing on the RHS is a MG for $0 \leqslant s \leqslant T$:

$$\mathbb{E}\left(\int_s^t e^{i\xi X_u}\,\mathrm{d}X_u \;\middle|\; \mathscr{F}_s\right) = 0 \quad \text{a.s.}$$

A priori we only know that $\int_0^t \mathbb{E}e^{i\xi X_u}\,\mathrm{d}X_u \in \mathscr{M}^2_{\mathrm{loc}}$. But for every reducing sequence τ_n for X and all fixed T:

$$\mathbb{E}\sup_{r\leqslant T}\left|X_t^{\tau_n}\right|^2 \overset{\text{Doob}}{\leqslant} \sup_{r\leqslant T}\mathbb{E}\,\langle X\rangle_r^{\tau_n} = 4\sup_{r\leqslant T}\mathbb{E}(r\wedge\tau_n) \leqslant 4T$$

and using Beppo Levi on the LHS ($n \to \infty$) and Theorem 2.14, we have $X^T \in \mathscr{M}^2_0$. Thus the stochastic integral is indeed a MG.

Next, we multiply (7.4) with $e^{-i\xi X_s}\mathbb{1}_F$, where $F \in \mathscr{F}_s$, and take expectations:

$$\mathbb{E}\left[e^{i\xi(X_t - X_s)}\mathbb{1}_F\right] - \mathbb{P}(F) = 0 - \frac{\xi^2}{2}\int_s^t \mathbb{E}\left[e^{i\xi(X_u - X_s)}\mathbb{1}_F\right]\mathrm{d}u. \tag{7.5}$$

This is an integral equality for the deterministic function $u \mapsto \mathbb{E}\left[e^{i\xi(X_u - X_s)}\mathbb{1}_F\right]$ that we can solve easily ($\rightsquigarrow$ Theorem 13.6). The unique solution is given by

$$\mathbb{E}\left[e^{i\xi(X_t - X_s)}\mathbb{1}_F\right] = \mathbb{P}(F)\,e^{-\frac{1}{2}(t-s)\xi^2},$$

thus we have shown (7.3). ■

Chapter 8

Lévy Processes and Poisson Random Measures

Let $(\Omega, \mathscr{A}, \mathbb{P})$ a complete probability space. First, we recall the definition of a Lévy process with values in $\mathbb{R}^d$, denoted $\mathrm{LP}(d)$, cf. lecture Lévy.

Definition 8.1. A process $\mathsf{L} = (L_t)_{t\geqslant 0}$ with values in $\mathbb{R}^d$ and

$$L_0 = 0 \text{ a.s.}; \tag{L_0}$$

$$L_{t_n} - L_{t_{n-1}}, \dots, Z_{t_1} - Z_{t_0} \perp\!\!\!\perp \qquad \forall n \in \mathbb{N},\ 0 = t_0 \leqslant t_1 < t_2 < \dots < t_n < \infty; \tag{L_1}$$

$$L_{t+s} - L_t \sim L_s \qquad \forall s, t \geqslant 0; \tag{L_2}$$

$$\lim_{s\to t} \mathbb{P}\big(|L_t - L_s| > \varepsilon\big) = 0 \qquad \forall \varepsilon > 0,\ t \geqslant 0. \tag{L_3}$$

is called **(d-dimensional) Lévy process**, denoted $\mathrm{LP}(d)$. By

$$\mathscr{F}_t^{\mathsf{L}} := \sigma(L_s : s \leqslant t) \tag{8.1}$$

we denote the **filtration generated by** L.

Remark 8.2. (a) In Lévy § 3 we have seen that

$$(L_0) - (L_3) \quad \Longleftrightarrow \quad \begin{cases} (L_0) & \\ (L_1') & L_t - L_s \perp\!\!\!\perp \mathscr{F}_s^{\mathsf{L}} \quad \forall 0 < s < t \\ (L_2) & \\ (L_3+) & t \mapsto L_t \quad \text{is } \mathbb{P}\text{-a.s. càdlàg.} \end{cases}$$

(b) It is possible to show, cf. [Pro05, Theorem I.31], that the filtration

$$\mathscr{F}_t' := \sigma(\mathscr{F}_t^{\mathsf{L}}, \mathscr{N}) \qquad \text{with} \quad \mathscr{N} := \{N \in \mathscr{F} : \quad \mathbb{P}(N) = 0\}$$

is **right continuous**, i.e. $\mathscr{F}_t' = \mathscr{F}_{t+}' = \bigcap_{h>0} \mathscr{F}_{t+h}'$.

Lemma 8.3. *Let* L *be an* $\mathrm{LP}(d)$. *Then*

$$L_t - L_s \quad \perp\!\!\!\perp \quad \mathscr{F}_{s+}^{\mathsf{L}} = \bigcap_{h>0} \mathscr{F}_{t+h}^{\mathsf{L}}.$$

Proof. We must show that

$$\mathbb{P}\big(\{L_t - L_s \in B\} \cap F\big) = \mathbb{P}\big(L_t - L_s \in B\big)\,\mathbb{P}(F)$$

for all $B \in \mathscr{B}(\mathbb{R}^d)$ and $F \in \mathscr{F}_{s+}^{\mathsf{L}}$. But for all $\xi \in \mathbb{R}^d$

$$\begin{aligned}
\mathbb{E}\big[e^{i(L_t - L_s)\xi}\mathbb{1}_F\big] &\overset{(L_3)}{=} \lim_{h\to 0} \mathbb{E}\big[e^{i(L_t - L_{s+h})\xi}\mathbb{1}_F\big] \\
&\underset{(L_2')}{\overset{\mathscr{F}_{s+}^{\mathsf{L}} \subset \mathscr{F}_{s+h}^{\mathsf{L}}}{=}} \lim_{h\to 0} \mathbb{E}\big[e^{i(L_t - L_{s+h})\xi}\big]\,\mathbb{E}\mathbb{1}_F \\
&= \mathbb{E}e^{i(L_t - L_s)\xi}\,\mathbb{P}(F).
\end{aligned}$$

■

Often, it is helpful to work with a larger filtration as $\mathscr{F}_t^{\mathsf{L}}$.

Definition 8.4. Let $(\mathscr{F}_t)_{t\geqslant 0}$ be a filtration. An $\mathscr{F}_t$-$\mathrm{LP}(d)$ is a stochastic process $\mathsf{L} = (L_t)_{t\geqslant 0}$ with (L_0), (L_2), $(L_3{+})$ such that

$$L_t \in \mathscr{F}_t \qquad \text{and} \qquad L_t - L_s \perp\!\!\!\perp \mathscr{F}_s \quad \forall 0 < s < t. \tag{L_1-$\mathscr{F}_t$}$$

Remark 8.5. Clearly,

$$\begin{aligned}
\mathsf{L} \text{ is } \mathrm{LP}(d) &\iff \mathsf{L} \text{ is } \mathscr{F}_t^{\mathsf{L}}\text{-}\mathrm{LP}(d) \\
&\implies \begin{cases} \mathsf{L} \text{ is } \mathscr{F}_{t+}^{\mathsf{L}}\text{-}\mathrm{LP}(d) \\ \mathsf{L} \text{ is } \mathscr{F}_t'\text{-}\mathrm{LP}(d) \end{cases} \\
\mathsf{L} \text{ is } \mathscr{F}_t\text{-}\mathrm{LP}(d) &\implies \mathsf{L} \text{ is } \mathscr{F}_t^{\mathsf{L}}\text{-}\mathrm{LP}(d) \implies \mathsf{L} \text{ is } \mathrm{LP}(d).
\end{aligned}$$

Remark 8.6. Let L be an $\mathscr{F}_t$-$\mathrm{LP}(d)$. If $\mathbb{E}\left|L_1\right| < \infty$, then

$$\mathbb{E}L_t = t\,\mathbb{E}L_1 =: tm \in \mathbb{R}^n \qquad \text{and} \qquad M_t := L_t - tm$$

is an $\mathscr{F}_t$-MG. If $\mathbb{E}\left|L_1\right|^2 < \infty$, then

$$\mathrm{Cov}\left(L_t^j, L_t^k\right) = t\,\mathrm{Cov}\left(L_1^j, L_1^k\right) =: tv_{jk} \in \mathbb{R}$$

and we get

$$M^j \in \mathscr{M}_0^2 \qquad \text{and} \qquad \left\langle M^j, M^k \right\rangle_t = tv_{jk}.$$

Indeed: By polarisation, it suffices to consider the case $d = 1$. Then for all $s \leqslant t$, and with $v := \mathbb{V}L_1 = \mathbb{E}\left[L_1 - m\right]^2$,

$$\begin{aligned}
\mathbb{E}\left(M_t^2 - tv \mid \mathscr{F}_s\right) &= \mathbb{E}\left(M_t^2 - M_s^2 \mid \mathscr{F}_s\right) + (M_s^2 - sv) + (s-t)v \\
&\overset{(3.4)}{\underset{(L_1\text{-}\mathscr{F}_t)}{=}} \underbrace{\mathbb{E}\left((M_t - M_s)^2 \mid \mathscr{F}_s\right) + (s-t)v}_{=\mathbb{E}M_{t-s}^2 + (s-t)v = (t-s)v + (s-t)v} + (M_s^2 - sv) \\
&= M_s^2 - sv.
\end{aligned}$$

Since $t \mapsto tv$ is continuous and thus $\mathscr{P}$-measurable, the claim follows.

Example 8.7. The following processes are Lévy processes:

(a) A Brownian motion plus drift, i.e. all LP with continuous paths, cf. Lévy §6.7.

(b) Poisson and compound Poisson processes.

Next, we want to go one step further and introduce so called *Poisson random measures* **(PRM)**.

Definition 8.8. (a) Let $(E, \mathscr{B})$ an arbitrary measurable space. A **point function** is a map $p : \mathbb{D}_p \to E$, where $\mathbb{D}_p \subset (0, \infty)$ is at most countable. By Π_E we denote the set of all E-valued point functions.

(b) The **counting measure associated to a point function** p is the counting measure

$$\begin{aligned} N_p((s,t] \times B) &:= \sharp \left\{ r \in \mathbb{D}_p : \quad (r, p(r)) \in (s,t] \times B \right\} \\ &= \sum_{r \in \mathscr{D}_p} \delta_{(r,p(r))}((s,t] \times B) \end{aligned} \tag{8.2}$$

defined on $((0,\infty) \times E, \mathscr{B}(0,\infty) \otimes \mathscr{B})$.

(c) A **point process** is a measurable map $p : \Omega \to \Pi_E$, $\omega \mapsto p(\omega) \in \Pi_E$.[a] The associated counting measure $N(\mathrm{d}s, \mathrm{d}z) := N_{p(\omega)}(\mathrm{d}s, \mathrm{d}z)$ is called **random measure**.

(d) $N_{p(\omega)}(\mathrm{d}s, \mathrm{d}z)$ is called **$\mathscr{F}_t$-Poisson random measure ($\mathscr{F}_t$-PRM)** and $p(\omega)$ is called **$\mathscr{F}_t$-Poisson point process ($\mathscr{F}_t$-PPP)** if

$$X_t := \big(N((0,t] \times B_1), \dots, N((0,t] \times B_m) \big), \qquad t \geqslant 0,$$

for all $m \in \mathbb{N}$ and all disjoint $B_1, \dots, B_m \in \mathscr{B}$, is an $\mathscr{F}_t$-LP(m).

[a]More precisely: We want that $\Omega \ni \omega \mapsto N_{p(\omega)}((0,t] \times U) \in \mathbb{R}$ for every $t > 0$ and $U \in \mathscr{B}(U)$ is a measurable map.

Theorem 8.9. *Let $N(\mathrm{d}s, \mathrm{d}z)$ be an $\mathscr{F}_t$-PRM. We set $N_t(B) := N((0,t] \times B)$. If $N_t(B) < \infty$ a.s. for all $t \geqslant 0$, then $\mathbb{E} N_t(B) < \infty$ and $N_t(B) \sim \mathrm{Poi}$ with intensity*

$$\mathbb{E} N_t(B) = t \, \mathbb{E} N_1(B) = t \, \nu(B)$$

and ν is a measure on $(E, \mathscr{B})$.

Before we prove the theorem, let us recall some statements about deterministic BV functions.

Remark 8.10 (Scholium). Let $f, g : [0, \infty) \to \mathbb{R}$ be càdlàg BV functions (WLOG even f, g increasing, if not we can decompose BV functions into differences of increasing functions). Then

(a) $f \in \mathrm{BV}$, $g \in \mathrm{BV} \cap \mathrm{C} \quad \Longrightarrow \quad [f, g] \equiv 0$.

(b) If $f(t) = \sum_{s \leqslant t} \Delta f(s)$, then $\int g(s-) \mathrm{d} f(s) = \sum_{s \leqslant t} g(s-) \Delta f(s)$.

(c) $f \in \mathrm{BV}$, $\varphi \in \mathrm{C}_b^1 \quad \Longrightarrow \quad \varphi \circ f \in \mathrm{BV}$.

Proof. (a) Fix $t > 0$ and write π for a generic partition of $[0, t]$. Then

$$\begin{aligned} [f, g]_t &= \lim_{|\pi| \to 0} \sum_{t_j \in \pi} \big(f(t_{j+1}) - f(t_j) \big) \big(g(t_{j+1}) - g(t_j) \big) \\ &\leqslant \lim_{|\pi| \to 0} \max_{t_j \in \pi} \big| f(t_{j+1}) - f(t_j) \big| \, \mathrm{var}(g; [0, t]) \xrightarrow{|\pi| \to 0} 0. \end{aligned}$$

(b) By definition of the Lebesgue-Stieltjes integral

$$\int g(s-)\mathrm{d}f(s) = \int g(s-)\mu_f(\mathrm{d}s),$$

where $\mu_f(a,b] := f(b) - f(a) = \sum_{a<s\leqslant b} \Delta f(s)$, i.e.

$$\mu_f(\mathrm{d}s) = \sum_s \Delta f \cdot \delta_s$$

and from this the claim follows.

(c) The assertion follows immediately by the inequality

$$|\varphi(f(t)) - \varphi(f(s))| \leqslant \|\varphi'\|_\infty |f(t) - f(s)|$$

and the definition of BV functions. ■

Proof (of Theorem 8.9). By assumption $L_t = N_t(B) < \infty$ is an LP(d). Then

$$\varphi_t(\xi) = \mathbb{E}e^{i\xi L_t} = e^{-t\psi(\xi)}$$

(cf. lecture LÉVY) and we can define

$$M_t^\xi := e^{i\xi L_t}e^{t\psi(\xi)}.$$

Obviously, $M_t^\xi \in \mathrm{L}^\infty(\mathbb{P})$ and $\left|M_t^\xi\right| = e^{t\mathrm{Re}\psi(\xi)} \geqslant 1$. Thus $M_{t\wedge T}^\xi \in \mathscr{M}^2$ for every fixed $T > 0$ – ok: complex valued, but by linearity we can consider Re and Im in all definitions – and

$$Y_t^\xi := \int_0^t \frac{1}{M_{s-}^\xi}\,\mathrm{d}M_s^\xi$$

is well-defined.

Clearly, Itô's lemma carries over verbatim to the complex valued case (consider Re and Im if necessary). We find

$$\begin{aligned}
\mathrm{d}M_s^\xi &= \mathrm{d}\left(e^{i\xi L_s}e^{s\psi(\xi)}\right) \\
&\overset{\text{Itô}}{\underset{\text{or (6.3)}}{=}} e^{s\psi(\xi)}\mathrm{d}e^{i\xi L_s} + e^{i\xi L_{s-}}\psi(\xi)e^{s\psi(\xi)}\mathrm{d}s + \underbrace{\mathrm{d}\left\langle e^{i\xi L}, e^{\bullet\psi(\xi)}\right\rangle_s}_{=0,\text{ as } [\mathrm{BV},\mathrm{C}_b\cap\mathrm{BV}]=0} \\
&= e^{-i\xi L_{s-}}M_{s-}^\xi\mathrm{d}e^{i\xi L_s} + \psi(\xi)M_{s-}^\xi\mathrm{d}s.
\end{aligned}$$

Thus

$$\begin{aligned}
Y_t^\xi &= \int_0^t e^{-i\xi L_{s-}}\mathrm{d}e^{i\xi L_s} + \int_0^t \psi(\xi)\mathrm{d}s \\
&\overset{8.10}{=} \sum_{0<s\leqslant t} \frac{\Delta e^{i\xi L_s}}{e^{i\xi L_{s-}}} + t\,\psi(\xi) \\
&= \sum_{0<s\leqslant t}\left(e^{i\xi\Delta L_s} - 1\right) + t\,\psi(\xi) \\
&= \begin{cases} \displaystyle\iint_B \left(e^{i\xi z} - 1\right) N_t(\mathrm{d}z) + t\,\psi(\xi) & \text{since } 8.10 \\ \left(e^{i\xi} - 1\right) N_t(B) + t\,\psi(\xi) & \text{since } \Delta L_{s-} = 1. \end{cases}
\end{aligned} \tag{8.3}$$

Because $\mathbb{E}Y_t^\xi = 0$, we get

$$
\begin{aligned}
&& \left(e^{i\xi}-1\right)\mathbb{E}N_t(B) &= -t\,\psi(\xi)\\
&\overset{\nu=\nu_1}{\Longrightarrow} & \nu_t(B) &= t\,\nu(B)\\
&\text{and also} & \mathbb{E}e^{i\xi L_t} &= e^{-t\,\psi(\xi)} = e^{t(e^{i\xi}-1)\nu(B)}\\
&\Longrightarrow & L_t &\sim \mathrm{Poi}(t\,\nu(B)).
\end{aligned}
$$

Since $\nu(B) = \mathbb{E}N_1(B)$, by Fubini's theorem, it follows that $C \mapsto \nu(B\cap C)$ is a measure. ■

Corollary 8.11. *Let N_t and ν as in Theorem 8.9 and $B_1, \dots, B_n \in \mathscr{B}$ pairwise disjoint sets. If $\nu(B_j) < \infty$, then the* LP *$N_t(B_j)$ are independent, $j = 1, \dots, n$.*

Proof. Note

$$\varphi_t(\xi) = \mathbb{E}e^{i\sum_1^n \xi_j N_t(B_j)} = e^{-t\psi(\xi)},$$

since the vector $(N_t(B_1), \dots, N_t(B_n))$ is an LP, and set

$$M_t^\xi = e^{i\sum_1^n \xi_j N_t(B_j)} e^{t\,\psi(\xi)}.$$

Since the $B_1, \dots, B_n$ are disjoint, the LP cannot have joint jumps and the analogue to the calculations in the proof of Theorem 8.9 shows – the key ingredient here is (8.3) –

$$
\begin{aligned}
Y_t^\xi &= \sum_{j=1}^n \left(e^{i\xi_j}-1\right) N_t(B_j) + t\,\psi(\xi)\\
\overset{t=1}{\Longrightarrow}\quad \sum_{j=1}^n \left(e^{i\xi_j}-1\right)\nu(B_j) &= -t\,\psi(\xi)\\
\Longrightarrow\quad \varphi_t(\xi) &= e^{t\sum_1^n (e^{i\xi_j}-1)\nu(B_j)}\\
&= \prod_1^n e^{t(e^{i\xi_j}-1)\nu(B_j)}\\
&= \prod_1^n e^{i\xi_j N_t(B_j)}.
\end{aligned}
$$

Because this relation holds for all $\xi_1, \dots, \xi_n \in \mathbb{R}$ it follows by Kac' theorem that the RV $N_1(B_j)$ are independent and thus also the LP $(N_t(B_j))_{t\geqslant 0}$ are independent (cf. lecture Lévy Corollary 3.6.). ■

Definition 8.12. Let $N(\mathrm{d}s, \mathrm{d}z)$ be an $\mathscr{F}_t$-PRM. Then we call

$$\hat{N}(\mathrm{d}s, \mathrm{d}z) := \mathrm{d}s\,\nu(\mathrm{d}z) \tag{8.4}$$

the **compensator of** N and

$$\widetilde{N}(\mathrm{d}s, \mathrm{d}z) := N(\mathrm{d}s, \mathrm{d}z) - \hat{N}(\mathrm{d}s, \mathrm{d}z) = N(\mathrm{d}s, \mathrm{d}z) - \mathrm{d}s\,\nu(\mathrm{d}z) \tag{8.5}$$

the **compensated PRM**.

Corollary 8.13. *Let $B \in \mathscr{B}$ with $\nu(B) < \infty$. Then*

$$\tilde{N}_t := \tilde{N}((0,t] \times B)$$

is an $\mathscr{M}^2$-MG and $\left\langle \tilde{N}_\bullet(B) \right\rangle_t = t\,\nu(B)$. Further,

$$\left\langle \tilde{N}_\bullet(B), \tilde{N}_\bullet(B') \right\rangle_t = t\,\nu(B \cap B')$$

for $B' \in \mathscr{B}$ with $\nu(B') < \infty$.

Proof. $M_t := \tilde{N}_t(B) = N_t(B) - t\,\nu(B)$ is a MG, since $N_t(B)$ is a PP with intensity $t\,\nu(B)$. Also

$$M_t^2 - t\,\nu(B) = (N_t(B) - t\,\nu(B))^2 - t\,\nu(B)$$

is a MG, cf. Remark 8.6, i.e. $\langle M \rangle_t = t\,\nu(B)$ by uniqueness of the angle bracket.

By polarisation we see that

$$\begin{aligned} \left\langle \tilde{N}_\bullet(B), \tilde{N}_\bullet(B') \right\rangle &= \left\langle \tilde{N}_\bullet(B \cap B') + N_\bullet(B \setminus B'), \tilde{N}_\bullet(B \cap B') + \tilde{N}_\bullet(B \setminus B') \right\rangle \\ &= \left\langle \tilde{N}_\bullet(B \cap B'), \tilde{N}_\bullet(B \cap B') \right\rangle \qquad (8.6) \\ &= t\,\nu(B \cap B'). \qquad (8.7) \end{aligned}$$

To see (8.6), note that

$$\begin{aligned} F \cap G = \emptyset \quad &\Longrightarrow \quad \tilde{N}_\bullet(F) \perp \tilde{N}_\bullet(G) \\ &\Longrightarrow \quad [\bullet, \bullet] = 0 \\ &\Longrightarrow \quad \langle \bullet, \bullet \rangle = 0. \end{aligned}$$

■

Corollary 8.14. *Let N, ν as above and $\varphi, \psi : E \to \mathbb{R}$ measurable functions with*

$$\int_E \left(|\varphi(z)|^2 + |\psi(z)|^2\right) \nu(\mathrm{d}z) < \infty.$$

Then

$$X_t^\varphi := \int \varphi(z)\tilde{N}_t(\mathrm{d}z), \qquad X_t^\psi := \int \psi(z)\tilde{N}_t(\mathrm{d}z),$$

are $\mathscr{M}_0^2$-MG and we have

$$\langle X^\varphi, X^\psi \rangle_t = t \int \varphi(z)\psi(z)\nu(\mathrm{d}z).$$

Proof. For elementary functions of the form

$$\varphi := \sum_j c_j \mathbb{1}_{B_j}, \qquad \psi := \sum_j d_j \mathbb{1}_{B_j},$$

where $B_1, B_2, \dots$ are disjoint with $\nu(B_j) < \infty$ – WLOG let B_j chosen such that they coincide for φ and ψ –, we find

$$\begin{aligned}\langle X^\varphi, X^\psi\rangle_t &\overset{(8.13)}{=} \left\langle \sum_j c_j \tilde{N}_\bullet(B_j), \sum_k d_k \tilde{N}_\bullet(B_k)\right\rangle_t \\ &= \sum_{j,k} c_j d_k \left\langle \tilde{N}_\bullet(B_j), \tilde{N}_\bullet(B_k)\right\rangle_t \\ &= \sum_{j,k} c_j d_k t\, \nu(B_j \cap B_k) \qquad (8.8)\\ &= \sum_j c_j d_j t\, \nu(B_j) \\ &= t \int \varphi\psi \,\mathrm{d}\nu. \qquad (8.9)\end{aligned}$$

Further, $X^\varphi = \sum_j c_j \tilde{N}_\bullet(B_j) \in \sum_{\text{finite}} \mathcal{M}^2 \subset \mathcal{M}^2$ and hence (8.8) follows by L^2 density of elementary functions for all $\varphi, \psi \in \mathsf{L}^2(\nu)$. Moreover,

$$\langle X^\varphi - X^{\varphi_n}\rangle_t^T = (t \wedge T) \int (\varphi - \varphi_n)\mathrm{d}\nu,$$

i.e. $X^{\varphi_n}_{\bullet\wedge T} \xrightarrow{n\to\infty} X^\varphi_{\bullet\wedge T} \in \mathcal{M}^2_0$, and hence X^φ is a MG. ■

Chapter 9

Stochastic Integrals and compensated PRM

We now introduce stochastic integrals where the integrators may be (compensated) PRMs. Therefore, we need some additional notions.

Definition 9.1. (a) Let N be a PRM with compensator $\hat{N}(\mathrm{d}s, \mathrm{d}z) = \mathrm{d}s\, \nu(\mathrm{d}z)$. By $\mathscr{E}(\tilde{N})$ we denote the set of all simple (previsible) processes of the form

$$g(\omega, s, z) := \sum_{j \text{ finite}} \psi_j(\omega, z) \mathbb{1}_{(s_j, s_{j+1}]}(s), \tag{9.1}$$

where $\psi_j \in \mathrm{L}^2(\mathscr{F}_{s_j} \otimes \mathscr{B}, \nu \otimes \mathscr{P})$, i.e. $\psi \in \mathscr{F}_{s_j} \otimes \mathscr{B}$ and satisfies

$$\mathbb{E} \int \left|\psi_j(z)\right|^2 \nu(\mathrm{d}z) = \|\psi\|^2_{\mathrm{L}^2(\nu \otimes \mathbb{P})} < \infty. \tag{9.2}$$

(b) A process $g(\omega, s, z)$, for $(\omega, s, z) \in \Omega \times [0, \infty) \times E$, is called **previsible (predictable)** if it is $\mathscr{P} \otimes \mathscr{B}$-measurable. We write $\mathrm{L}^2(\tilde{N}) \cap \mathscr{B}(\mathscr{P} \otimes \mathscr{B})$ for the previsible processes with

$$\mathbb{E} \int_0^\infty \int_E |g(s, z)|^2 \, \hat{N}(\mathrm{d}s, \mathrm{d}z) = \|g\|^2_{\mathrm{L}^2(\nu \otimes \mathbb{P} \otimes \lambda)} < \infty. \tag{9.3}$$

The (slightly modified) modified proof of Theorem 3.13 (i.e. $\overline{\mathscr{E}} = \mathrm{L}^2(\langle X \rangle) \cap \mathscr{B}(\mathscr{P})$) shows that processes of the form

$$\begin{cases} \sum_{j \text{ finite}} \psi_j(\omega) \mathbb{1}_{B_j}(z) \mathbb{1}_{(s_j, s_{j+1}]}(s) \\ 0 = s_0 < s_1 < \ldots < \ldots, \; B_j \in \mathscr{B}(E) \text{ disjoint}, \; \nu(B_j) < \infty, \; \psi_j \in \mathrm{L}^2(\mathscr{F}_{s_j}, \mathbb{P}) \end{cases} \tag{9.4}$$

are $\mathrm{L}^2(\mathbb{P} \otimes \mu \otimes \lambda)$-dense in $\mathrm{L}^2(\tilde{N}) \cap \mathscr{B}(\mathscr{P} \otimes \mathscr{B})$. In particular also

Proposition 9.2. $\overline{\mathscr{E}(\tilde{N})}^{\mathrm{L}^2(\mathbb{P} \otimes \mu \otimes \lambda)} = \mathrm{L}^2(\tilde{N}) \cap \mathscr{B}(\mathscr{P} \otimes \mathscr{B})$.

To define the stochastic integral $\iint_{(0,\infty) \times E} g(s, z) \hat{N}(\mathrm{d}s, \mathrm{d}z)$, we start again by step functions. By Corollary 8.13,

$$\mathbb{E}\left[\tilde{N}((s, t] \times B_1)\, \tilde{N}((s, t] \times B_2)\right] \overset{\text{compensator}}{=} \hat{N}((s, t] \times (B_1 \cap B_2)) = (t - s)\nu(B_1 \cap B_2). \tag{9.5}$$

Since step functions $\sum_j \alpha_j \mathbb{1}_{A_j}(\omega) \mathbb{1}_{B_j}(\omega)$, $A_j \in \mathscr{F}_s$, $B_j \in \mathscr{B}$ with $\nu(B_j) < \infty$, are L^2-dense in the space of $\mathscr{F}_s$-measurable RV $\psi(s, z)$ such that $\mathbb{E} \int_E |\psi(z)|^2 \, \nu(\mathrm{d}z) < \infty$ (use the Sombrero lemma for the measure $\mathbb{P} \otimes \nu$), it follows by (9.5) that

$$\mathbb{E}\left(\left[\int_E \psi(z) \tilde{N}((s, t], \mathrm{d}z) \right]^2 \,\middle|\, \mathscr{F}_s \right) \quad = \quad \sum_{j,k} \mathbb{E}\left(\alpha_j \alpha_k \mathbb{1}_{A_j} \mathbb{1}_{A_k} \tilde{N}((s, t], B_j)\, \tilde{N}((s, t], B_k) \,\middle|\, \mathscr{F}_s \right)$$

$$
\begin{aligned}
&\overset{\text{pull out}}{=} \sum_{j,k} \alpha_j \alpha_k \mathbb{1}_{A_j} \mathbb{1}_{A_k} \mathbb{E}\Big(\tilde{N}((s,t],B_j)\,\tilde{N}((s,t],B_k) \;\Big|\; \mathscr{F}_s\Big) \\
&\overset{(9.5)}{=} \sum_{j,k} \alpha_j \alpha_k \mathbb{1}_{A_j} \mathbb{1}_{A_k} (t-s)\nu(B_j \cap B_k) \text{ (9.6)} \\
&= \sum_j \alpha_j^2 \mathbb{1}_{A_j} (t-s)\nu(B_j) \\
&= (t-s) \int_E \psi^2(z)\nu(\mathrm{d}z) \\
&= \int_s^t \int_E \psi^2(z)\nu(\mathrm{d}z)\mathrm{d}r.
\end{aligned}
$$

Definition 9.3. For $\psi_j(z) \in \mathrm{L}^2(\mathbb{P} \otimes \nu) \cap \mathscr{B}(\mathscr{F}_{s_j} \otimes \mathscr{B})$, let $g(\omega, s, z)$ be a stochastic process of the form $g(\omega, s, z) = \sum_{j \text{ finite}} \psi_j(z) \mathbb{1}_{(s_j, s_{j+1}]}(s)$ (so in particular: $g \in \mathrm{L}^2(\tilde{N}) \cap \mathscr{B}(\mathscr{P} \otimes \mathscr{B})$). Then

$$
\begin{aligned}
X_t &:= \sum_j \int \psi_j(z)\tilde{N}((s_j \wedge t, s_{j+1} \wedge t], \mathrm{d}z) \\
&= \sum_j \int \psi_j(z)\Big(\underbrace{\tilde{N}(s_{j+1} \wedge t, \mathrm{d}z)}_{=:\, \tilde{N}((0,s_{j+1}\wedge t],\mathrm{d}z)} - \tilde{N}(s_j \wedge t, \mathrm{d}z)\Big)
\end{aligned} \tag{9.7}
$$

is the **stochastic integral of g with respect to the compensated PRM $\tilde{N}$**. We denote the process by

$$
\int_0^t \int_E g(s,z)\widetilde{N}(\mathrm{d}s, \mathrm{d}z).
$$

Lemma 9.4. *For the PRM N and $g \in \mathscr{E}(\tilde{N}) \subset \mathrm{L}^2(\tilde{N}) \cap \mathscr{B}(\mathscr{P} \otimes \mathscr{B})$, it holds that $\int_0^t \int_E g(s,z)\widetilde{N}(\mathrm{d}s, \mathrm{d}z) \in \mathscr{M}_0^2$ and*

$$
\left\langle \int_0^\bullet \int_E g(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \right\rangle_t = \int_0^t \int_E |g(s,z)|^2 \,\hat{N}(\mathrm{d}s,\mathrm{d}z), \tag{9.8}
$$

$$
\mathbb{E}\left|\int_0^t \int_E g(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right|^2 = \mathbb{E}\int_0^t \int_E |g(s,z)|^2 \,\hat{N}(\mathrm{d}s,\mathrm{d}z). \tag{9.9}
$$

Proof. WLOG let $s = s_m < s_n = t$, if not we add $s < t$ to the partition in (9.1). By Corollary 8.14, $X \in \mathscr{M}_0^2$. Further,

$$
\begin{aligned}
\mathbb{E}\big((X_t - X_s)^2 \mid \mathscr{F}_s\big) &= \sum_{j=m}^{n-1} \mathbb{E}\Big((X_{s_{j+1}} - X_{s_j})^2 \;\Big|\; \mathscr{F}_s\Big) \\
&\qquad \big[\mathrm{L}^2\text{-MG, mixed terms} = 0, \text{ step } 3^\circ \text{ in cf. Proposition 3.8 (a)}\big] \\
&= \sum_{j=m}^{n-1} \mathbb{E}\left(\left|\int_E \psi_j(z)\tilde{N}((s_j, s_{j+1}], \mathrm{d}z)\right|^2 \;\middle|\; \mathscr{F}_s\right) \\
&\overset{\text{tower}}{=} \sum_{j=m}^{n-1} \mathbb{E}\left(\mathbb{E}\left(\left|\int_E \psi_j(z)\tilde{N}((s_j, s_{j+1}], \mathrm{d}z)\right|^2 \;\middle|\; \mathscr{F}_{s_j}\right) \;\middle|\; \mathscr{F}_s\right)
\end{aligned}
$$

$$\stackrel{(9.6)}{=} \sum_{j=m}^{n-1} \mathbb{E}\left(\int_E \left|\psi_j(z)\right|^2 \tilde{N}((s_j, s_{j+1}], \mathrm{d}z) \;\middle|\; \mathscr{F}_s \right)$$
$$= \mathbb{E}\left(\int_0^t \int_E |g(r,z)|^2 \hat{N}(\mathrm{d}r, \mathrm{d}z) \;\middle|\; \mathscr{F}_s \right).$$

Thus (9.9) follows by uniqueness of the angle bracket and (9.9) follows by (9.8) by taking expectations. ■

Remark 9.5. We can read formula (9.9) also as follows: For all $T \leqslant \infty$, it holds

$$\begin{aligned}
\left\| \int_0^\bullet \int_E |g(s,z)|\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \right\|^2_{\mathscr{M}_0^2[0,T]} &\stackrel{\text{def}}{=} \mathbb{E} \sup_{t \leqslant T} \left[\int_0^\bullet \int_E |g(s,z)|\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \right]^2 \\
&\underset{c=4}{\stackrel{\text{Doob}}{\leqslant}} c\, \mathbb{E} \left[\int_0^T \int_E |g(s,z)|\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \right]^2 \\
&\stackrel{(9.9)}{=} c\, \mathbb{E} \int_0^T \int_E |g(s,z)|^2\, \hat{N}(\mathrm{d}s, \mathrm{d}z) \\
&= c\, \|g\|^2_{\mathrm{L}^2(\tilde{N})}.
\end{aligned}$$

Since we can also estimate below with constant $c = 1$ in the step marked by «Doob», we find for $g \in \mathscr{E}(\tilde{N})$

$$\left\| \int_0^\bullet \int_E |g(s,z)|\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \right\|^2_{\mathscr{M}_0^2[0,T]} \asymp \|g\|^2_{\mathrm{L}^2(\tilde{N})} \tag{9.10}$$

with constant independent of T or g.

Therefore, we can also declare the stochastic integral for all $g \in \mathrm{L}^2(\tilde{N}) \cap \mathscr{B}(\mathscr{P} \otimes \mathscr{B})$ by means of the BLT principle (cf. § 3.9): For all $T \in (0, \infty]$, we get

$$\int_0^{\bullet \wedge T} \int_E g(s,z) \widetilde{N}(\mathrm{d}s, \mathrm{d}z) = \mathscr{M}_0^2 - \lim_{n \to \infty} \int_0^\bullet \int_E g_n(s,z) \widetilde{N}(\mathrm{d}s, \mathrm{d}z), \tag{9.11}$$

where $\mathscr{E}(\tilde{N}) \ni g_n^T \xrightarrow{\mathrm{L}^2(\tilde{N})} g^T$.

Mind This limit does not depend on the approximating sequence $(g_n)_n$!

Theorem 9.6. *Let $g^T \in \mathrm{L}^2(\tilde{N})$ for every $T > 0$. Then $\int_0^\bullet \int_E g(s,z) \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \in \mathscr{M}_0^2$ and the relations eqs. (9.8) and (9.9) still hold.*

Proof. Clear – by extension of the integral via the BLT principle and continuity of the norm. ■

Remark 9.7. Let $f(\omega, s, z)$ a $\mathscr{P} \otimes \mathscr{B}$-measurable process and

$$\mathbb{E} \int_0^T \int_E |f(s,z)|^2 \,\mathrm{d}s \nu(\mathrm{d}z) < \infty, \qquad T \leqslant \infty.$$

Then we can show, analogous to Corollary 8.14, Definition 9.1, and Theorem 9.6, that

$$\int_0^t \int_E f(s,z) N(\mathrm{d}s, \mathrm{d}z) \qquad \text{and} \qquad \int_0^t \int_E f(s,z) \hat{N}(\mathrm{d}s, \mathrm{d}z) \tag{9.12}$$

are well-defined, and that

$$\int_0^t \int_E f(s,z)N(\mathrm{d}s,\mathrm{d}z) - \int_0^t \int_E f(s,z)\hat{N}(\mathrm{d}s,\mathrm{d}z)$$

is a MG (**Note:** both terms exist separately!). In particular,

$$\mathbb{E}\int_0^t \int_E f(s,z)N(\mathrm{d}s,\mathrm{d}z) = \mathbb{E}\int_0^t \int_E f(s,z)\hat{N}(\mathrm{d}s,\mathrm{d}z)$$

and by

$$\begin{aligned} \left|\int_0^t \int_E f(s,z)N(\mathrm{d}s,\mathrm{d}z)\right| &\leqslant \mathbb{E}\int_0^t \int_E |f(s,z)|\, N(\mathrm{d}s,\mathrm{d}z) \\ &\overset{\text{MG}}{=} \mathbb{E}\int_0^t \int_E |f(s,z)|\, \hat{N}(\mathrm{d}s,\mathrm{d}z) < \infty, \end{aligned}$$

both processes in (9.12) are BV processes.

Chapter 10

The Itô lemma (part 2)

Next, we want to extend Itô's lemma in § 7, (7.1), to integrals driven by an $\mathscr{F}_t$-PRM $N(\mathrm{d}s, \mathrm{d}z)$.

Theorem 10.1. *Let $X_t = (X_t^1, \dots, X_t^d)$ a (vector of) $\mathscr{F}_t$-SMG, where*

$$\begin{cases} X_t^j = X_0^j + A_t^j + M_t^j + D_t^j \\ D_t^j = \int_0^t \int_E g^j(s,z) \widetilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_E h^j(s,z) N(\mathrm{d}s, \mathrm{d}z) \end{cases} \tag{10.1}$$

with $A^j \in \mathrm{BV} \cap \mathrm{C}$, $M^j \in \mathscr{M}_0^{c,\mathrm{loc}}$, N a PRM, $g \in \mathrm{L}^2(\tilde{N}) \cap \mathscr{B}(\mathscr{P} \otimes \mathscr{B})$ and $h \in \mathrm{L}^1(\hat{N}) \cap \mathscr{B}(\mathscr{P} \otimes \mathscr{B})$ with $|g| \cdot |h| = 0$.[a] For $F : \mathbb{R}^d \to \mathbb{R}$, $F \in \mathrm{C}^2$, it holds

$$\begin{aligned} F(X_t^1, \dots, X_t^d) &- F(X_0^1, \dots, X_0^d) \\ &= \sum_{j=1}^d \int_0^t \partial_j F(X_{s-}) \, \mathrm{d}A_s^j + \sum_{j=1}^d \int_0^t \partial_j F(X_{s-}) \, \mathrm{d}M_s^j \\ &+ \frac{1}{2} \sum_{j,k=1}^d \int_0^t \partial_j \partial_k F(X_{s-}) \, \mathrm{d} \left\langle M^j, M^k \right\rangle_s \\ &+ \int_0^t \int_E \big(F(X_{s-} + g(s,z)) - F(X_{s-}) \big) \, \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\ &+ \int_0^t \int_E \big(F(X_{s-} + h(s,z)) - F(X_{s-}) \big) \, N(\mathrm{d}s, \mathrm{d}z) \\ &+ \int_0^t \int_E \big(F(X_{s-} + g(s,z)) - F(X_{s-}) - \nabla F(X_{s-}) \cdot g(s,z) \big) \, \hat{N}(\mathrm{d}s, \mathrm{d}z). \end{aligned} \tag{10.2}$$

[a] With $|f|^2 = \sum_j |f_j|^2$ the Euclidean norm.

Before we give a proof, we need some more preliminary considerations on the integration of PRMs.

Remark 10.2 (Scholium). We use the notations of § 8 again. For a PRM $N(\mathrm{d}s, \mathrm{d}z) = N_p(\mathrm{d}s, \mathrm{d}z)$, given by the point function p with domain $\mathbb{D}_p$, we have

$$N_p(\mathrm{d}s, \mathrm{d}z) = \sum_{r \in \mathbb{D}_p} \delta_{(r, p(r))}(\mathrm{d}s, \mathrm{d}z).$$

Thus,

$$\iint f(s,z) N(\mathrm{d}s, \mathrm{d}z) = \sum_{r \in \mathbb{D}_p} f(r, p(r)).$$

The integral with respect to $\widetilde{N}(\mathrm{d}s, \mathrm{d}z)$ is defined as L^2-limit respectively.

In Theorem 10.1 it holds

$$D_t = \iint g(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) + \iint h(s,z)N(\mathrm{d}s,\mathrm{d}z)$$

und thus is (for $g \in \mathscr{E}(\tilde{N})$ obvious, else approximation):

$$\Delta D_t = \begin{cases} g(t,p(t)) + h(t,p(t)), & t \in \mathbb{D}_p \\ 0, & \text{else.} \end{cases}$$

Hence (since $|g| \cdot |h| \equiv 0$) for Φ with $\Phi(x,x) = 0$:

$$\begin{aligned}\sum_{s\leqslant t} \Phi(X_s, X_{s-}) &= \sum_{s\leqslant t,\, \Delta X_s \neq 0} \Phi(X_{s-} + \Delta X_{s-}, X_{s-}) \\ &= \sum_{r\in\mathbb{D}_p} \Phi(X_{r-} + g(r,p(r)), X_{r-}) + \sum_{r\in\mathbb{D}_p} \Phi(X_{r-} + h(r,p(r)), X_{r-}) \\ &= \iint \Phi(X_{s-} + g(s,z) + h(s,z), X_{s-})N(\mathrm{d}s,\mathrm{d}z).\end{aligned}$$

Proof (of Theorem 10.1). We use (7.1): Set $C_t := M_t + A_t$ for the purely continuous part of X_t. Then, for $F \in \mathrm{C}_b^2$,

$$\begin{aligned}F(X_t) - F(X_0) = &\sum_{j=1}^d \int_0^t \partial_j F(X_{s-})\,\mathrm{d}(A^j + M^j)_s \\ &+ \frac{1}{2}\sum_{j,k=1}^d \int_0^t \partial_j\partial_k F(X_{s-})\,\mathrm{d}\left[X^j, X^k\right]_s \\ &+ \sum_{j=1}^d \int_0^t \partial_j F(X_{s-})\,\mathrm{d}D_s^j \\ &+ \sum_{s\leqslant t}\left(F(X_s) - F(X_{s-}) - \nabla F(X_{s-})\cdot \Delta X_s\right).\end{aligned} \tag{10.3}$$

As $[\mathrm{BV},\mathrm{C}] \equiv 0$, cf. Remark 8.10, it follows

$$\left[X^j, X^k\right]_s^c \overset{6.4}{=} \left[C^j, C^k\right]_s \overset{8.10}{=} \left[M^j, M^k\right]_s \overset{6.4}{=} \left\langle M^j, M^k\right\rangle_s$$

and thus the first three terms in (10.3) and (10.2) coincide.

Finally, the purely jump terms: By Lemma 10.3 below,

$$\mathrm{d}\int_0^t\int_E g_j(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) = \int_E g_j(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \tag{10.4}$$

$$\mathrm{d}\int_0^t\int_E h_j(s,z)N(\mathrm{d}s,\mathrm{d}z) = \int_E h_j(s,z)N(\mathrm{d}s,\mathrm{d}z) \tag{10.5}$$

and thus

$$\int_0^t \partial_j F(X_{s-})\,\mathrm{d}D_s^j = \int_0^t\int_E \partial_j F(X_{s-})g_j(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) + \int_0^t\int_E \partial_j F(X_{s-})h_j(s,z)N(\mathrm{d}s,\mathrm{d}z)$$

as well as

$$\sum_{s\leqslant t}\big(F(X_s)-F(X_{s-})-\nabla F(X_{s-})\nabla X_s\big)$$

$$\begin{aligned}
&=\int_0^t\int_E\big(F(X_{s-}+g(s,z))-F(X_{s-})-\nabla F(X_{s-})g(s,z)\big)\,N(\mathrm{d}s,\mathrm{d}z)\\
&\qquad+\int_0^t\int_E\big(F(X_{s-}+g(s,z))-F(X_{s-})-\nabla F(X_{s-})h(s,z)\big)\,N(\mathrm{d}s,\mathrm{d}z)\\
&=\int_0^t\int_E\big(F(X_{s-}+g(s,z))-F(X_{s-})-\nabla F(X_{s-})g(s,z)\big)\,\widetilde N(\mathrm{d}s,\mathrm{d}z)\\
&\qquad+\int_0^t\int_E\big(F(X_{s-}+g(s,z))-F(X_{s-})-\nabla F(X_{s-})g(s,z)\big)\,\hat N(\mathrm{d}s,\mathrm{d}z)\\
&\qquad+\int_0^t\int_E\big(F(X_{s-}+g(s,z))-F(X_{s-})-\nabla F(X_{s-})h(s,z)\big)\,N(\mathrm{d}s,\mathrm{d}z).
\end{aligned}$$

Adding the last two terms, we immediately get (10.2) for $F\in C_b^2$.

If now $F\in C^2$, then we stop by

$$\tau_n := \{s>0 : |X_s|>n\}\wedge n$$

and mind that

$$\big|F(X_{s-}^{\tau_n})+\partial_j F(X_{s-}^{\tau_n})+\partial_j\partial_k F(X_{s-}^{\tau_n})\big|\leqslant\sum_{|\alpha|\leqslant 2}\sup_{|x|\leqslant n}|\partial^\alpha F(x)|<\infty.$$

Now we can localise the stochastic integrals over the upper bound (exercise!). ■

We only have to show eqs. (10.4) and (10.5).

Lemma 10.3. *Formulae (10.4) and (10.5) hold.*

Proof. We prove (10.4). We write for $F\in C_b^2$: $F(X_-)_s := F(X_{s-})$. Then

$$\begin{aligned}
\left\langle F(X_-)\bullet\int_0^\bullet\int_E f_j(r,z)\widetilde N(\mathrm{d}r,\mathrm{d}z)\right\rangle_t &\overset{5.5}{=}\int_0^t F^2(X_{s-})\,\mathrm{d}_s\left\langle\int_0^\bullet\int_E f_j(r,z)\widetilde N(\mathrm{d}r,\mathrm{d}z)\right\rangle_s\\
&\overset{9.4}{=}\int_0^t F^2(X_{s-})\,\mathrm{d}_s\underbrace{\int_0^s\int_E\big|f_j(r,z)\big|^2\nu(\mathrm{d}z)\mathrm{d}r}_{=\int_E|f_j(s,z)|^2\nu(\mathrm{d}z)\mathrm{d}s}\\
&=\int_E\big|F(X_{s-})f_j(s,z)\big|^2\nu(\mathrm{d}z)\mathrm{d}s\\
&\overset{9.4}{=}\left\langle\int_0^t\int_E F(X_{s-})f_j(s,z)\widetilde N(\mathrm{d}s,\mathrm{d}z)\right\rangle_\bullet .
\end{aligned}$$

Now, we consider

$$I := \left\langle F(X_-)\bullet\int_0^\bullet\int_E f_j(r,z)\widetilde N(\mathrm{d}r,\mathrm{d}z)-\int_0^t\int_E F(X_{s-})f_j(s,z)\widetilde N(\mathrm{d}s,\mathrm{d}z)\right\rangle_t .$$

«Expanding» this angle bracket, we find by the calculation above that

$$\begin{aligned}
I&=2\left\langle\int_0^t\int_E F(X_{s-})f_j(s,z)\widetilde N(\mathrm{d}s,\mathrm{d}z)\right\rangle_\bullet\\
&\qquad-2\left\langle F(X_-)\bullet\int_0^\bullet\int_E f_j(r,z)\widetilde N(\mathrm{d}r,\mathrm{d}z),\int_0^t\int_E F(X_{s-})f_j(s,z)\widetilde N(\mathrm{d}s,\mathrm{d}z)\right\rangle_t
\end{aligned}$$

and again by the equality above, we see that $I = 0$. The assertion (10.4) thus follows by uniqueness of the angle bracket.[1]

The equality (10.5) is easier to show, because the claim is about pathwise Stieltjes integrals. ■

We close this chapter with an application of Itô's formula.

Theorem 10.4. *Let* $Y \in \mathcal{M}^c$ *an* $\mathcal{F}_t$*-LP*(d) *with* $\mathbb{E}Y_t^2 < \infty$*. Then* Y_t *is Gaussian distributed and independent of every* $\mathcal{F}_t$*-PRM.*

Proof. For simplicity (and WLOG) let us consider only the case $d = 1$. Set $\mathbb{E}Y_1^2 = \sigma^2$. Our proof shows again the «Gaussian distribution», although already known from Theorem 7.3. Actually this theorem can be applied, since

$$Y_0 = 0 \quad \overset{\text{MG}}{\Longrightarrow} \quad \mathbb{E}Y_t = 0$$

and since

$$\begin{aligned} \mathbb{E}Y_t^2 \quad &\overset{\text{LP}}{\Longrightarrow} \quad \mathbb{E}Y_t^2 = t\,\sigma^2 \\ &\overset{\text{LP}}{\Longrightarrow} \quad \langle Y \rangle_t = t\,\sigma^2. \end{aligned}$$

Set $Z_t := N_t(B)$ where $\nu(B) < \infty$, and where $N(\mathrm{d}s, \mathrm{d}z)$ is an $\mathcal{F}_t$-PRM. Then Itô's formula shows that

$$\begin{aligned} U_t &:= e^{i\xi Y_t} e^{i\eta Z_t} \\ &= i\xi \int_0^t U_{s-}\,\mathrm{d}Y_s - \frac{1}{2}\xi^2 \int_0^t U_{s-}\sigma^2 \mathrm{d}s + \int_0^t \int_E U_{s-}(e^{i\eta} - 1)\mathbb{1}_B(z) N(\mathrm{d}s, \mathrm{d}z). \end{aligned}$$

Indeed:

$$\begin{aligned} (X_t^1, X_t^2) &= (M_t^1, D_t^2) = \left(Y_t, \int_0^t \int_E 0\tilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_E \mathbb{1}_B(z) N(\mathrm{d}s, \mathrm{d}z) \right), \\ F(x_1, x_2) &= e^{i\xi x_1} e^{i\eta x_2}, \end{aligned}$$

i.e. $g_2 \equiv 0$ and $h_2 \equiv \mathbb{1}_B$; so

$$\begin{aligned} F(X_{s-} + (0, h_2)) - F(X_{s-}) &= e^{i\xi Y_{s-}} e^{i\eta(Z_{s-} + h_2)} - e^{i\xi Y_{s-}} e^{i\eta Z_{s-}} \\ &= e^{i\xi Y_{s-}} e^{i\eta Z_{s-}} \underbrace{\left(e^{i\eta h_2} - 1\right)}_{= (e^{i\eta} - 1)\mathbb{1}_B(z)}. \end{aligned}$$

Taking expectations on both sides of the last equation, noting that $U_{s-}\mathbb{1}_B(z) \in \mathrm{L}^2(\tilde{N}) \cap \mathcal{B}(\mathcal{P} \otimes \mathcal{B})$, we get

$$\begin{aligned} \mathbb{E}U_t &= i\xi\,\mathbb{E} \int_0^t U_{s-}\,\mathrm{d}Y_s - \frac{1}{2}\xi^2 \int_0^t \mathbb{E}U_{s-}\sigma^2 \mathrm{d}s + \mathbb{E} \int_0^t \int_E U_{s-}(e^{i\eta} - 1)\mathbb{1}_B(z) N(\mathrm{d}s, \mathrm{d}z) \\ &\overset{9.6}{=} 0 - \frac{1}{2}\xi^2 \int_0^t \mathbb{E}U_s \sigma^2 \mathrm{d}s + \int_0^t \int_E \mathbb{E}U_s(e^{i\eta} - 1)\mathbb{1}_B(z)\nu(\mathrm{d}z)\mathrm{d}s \\ &= -\frac{\sigma^2\xi^2}{2} \int_0^t \mathbb{E}U_s \mathrm{d}s + \nu(B)(e^{i\eta} - 1) \int_0^t \mathbb{E}U_s \mathrm{d}s. \end{aligned}$$

[1]Note: If $M \in \mathcal{M}_0^2$ with $\langle M \rangle = 0$, then $\mathbb{E}M_t^2 = \mathbb{E}\langle M \rangle_t = 0$, i.e. $M = 0$.

Solving this ODE, we find

$$\mathbb{E}U_t = \exp\left[-t\left(\frac{\sigma^2\xi^2}{2} + (e^{i\eta}-1)\nu(B)\right)\right].$$

Hence,

$$\begin{aligned}\mathbb{E}e^{i\xi Y_t}e^{i\eta Z_t} &\overset{(*)}{=} e^{-t\sigma^2\xi^2/2}e^{-t(1-e^{i\eta})\nu(B)} \\ &= \underbrace{\mathbb{E}e^{i\xi Y_t}}_{\text{use } (*) \text{ with } \eta=0} \cdot \underbrace{\mathbb{E}e^{i\xi Z_t}}_{\text{use } (*) \text{ with } \xi=0}\end{aligned}$$

showing $Y_t \perp\!\!\!\perp Z_t$ by Kac' theorem and $Y_1 \sim \mathsf{N}(0,1)$. The independence of all processes now follows from the fact that we have given an LP (cf. lecture Lévy). ■

Chapter 11

Lévy processes (part 2)

Let $(Z_t)_{t\geqslant 0}$ be an $\mathscr{F}_t$-LP(m). Since Z_t has càdlàg paths, $\Delta Z_r := Z_r - Z_{r-}$ and

$$N((s,t]\times B) := \sharp\left\{r\in(s,t]\,:\,\Delta Z_r\in B\right\} \tag{11.1}$$

are well-defined and $\in[0,\infty]$. From the lecture Lévy (therein we showed that $N_t(B)$ are Poisson processes and $N_t(B_j)$ are disjoint for B_j disjoint) we know that $N(\mathrm{d}s,\mathrm{d}z)$ is a PRM. Note that $N_t(B) := N((0,t]\times B) < \infty$ a.s. if $0\notin\overline{B}$ (i.e. if $B\cap B_\delta(0)=\varnothing$ for some $\delta>0$), because càdlàg paths only allow finitely many jumps $\left|\Delta Z_r\right|>\delta$ for fixed $\delta>0$ and all $r\in(s,t]$. In particular, also

$$N(\mathrm{d}s,\mathrm{d}z) = \sum_{r:\,\Delta Z_r\neq 0}\delta_{(r,\Delta Z_r)}(\mathrm{d}s,\mathrm{d}z).$$

By Theorem 8.9, automatically $\nu(B)<\infty$. In particular,

$$\nu(B(0,\varepsilon)^c)<\infty \qquad \forall\varepsilon>0. \tag{11.2}$$

We will see how to represent the LP(m) Z_t as stochastic integrals.

Theorem 11.1 (Lévy-Itô decomposition). *Let $(Z_t)_{t\geqslant 0}$ be an $\mathscr{F}_t$-LP(m). Then*

$$Z_t = \sqrt{Q}B_t + \ell t + \int_0^t\int_{\{|z|>1\}} z\,N(\mathrm{d}s,\mathrm{d}z) + \int_0^t\int_{\{0<|z|\leqslant 1\}} z\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z), \tag{11.3}$$

where $\mathsf{B}=(B_t)_{t\geqslant 0}$ is an $(\mathscr{F}_t)_{t\geqslant 0}$-adapted BM$(m)$, $Q\geqslant 0$ a positive semidefinite $m\times m$-matrix, $\ell\in\mathbb{R}^m$ and $N(\mathrm{d}s,\mathrm{d}z)$ the PRM on $[0,\infty)\times(\mathbb{R}^m\setminus\{0\})$ induced by Z. In particular $N(\mathrm{d}s,\mathrm{d}z)\perp\!\!\!\perp B_t$ and the intensity measure $\hat{N}(\mathrm{d}s,\mathrm{d}z)=\mathrm{d}s\,\nu(\mathrm{d}z)$ satisfies the condition

$$\int_{z\neq 0}\frac{|z|^2}{1+|z|^2}\nu(\mathrm{d}z)<\infty. \tag{11.4}$$

The so called **Lévy-Itô representation** *(11.3) is unique in the following sense: If $B'_t, N'(\mathrm{d}s,\mathrm{d}z)$ are an $\mathscr{F}_t$-BM and -PRM, representing Z_t as in (11.3), then $\mathsf{B}=\mathsf{B}'$ and $\mathsf{N}=\mathsf{N}'$ (up to indistinguishability).*

Before proving Theorem 11.1, we state an immediately consequence.

Corollary 11.2 (Lévy-Khintchine formula). *The characteristic function $\psi(\xi)$ of an LP(m) Z_t is given by*

$$\mathbb{E}e^{i\xi Z_t} = e^{-t\psi(\xi)}, \tag{11.5}$$

where

$$\psi(\xi) = -i\ell\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \int_{z\neq 0}\left(1-e^{iz\cdot\xi}+iz\cdot\xi\mathbb{1}_{\{|z|\leqslant 1\}}\right)\nu(\mathrm{d}z). \tag{11.6}$$

Proof (of Theorem 11.1 and Corollary 11.2). We base the idea of the proof on Theorem 8.9. Since $(Z_t)_{t\geqslant 0}$ is infinitely divisible – cf. Lévy –, we have $\mathbb{E}e^{i\xi Z_t} = e^{-t\psi(\xi)}$. For simplicity (and WLOG), we assume $d = 1$. We set

$$M_t^\xi = e^{i\xi Z_t} e^{t\psi(\xi)}, \qquad \xi \in \mathbb{R}^m.$$

Then $M_t^\xi \in \mathrm{L}^\infty(\mathscr{P})$ and M^ξ is a MG. Further, let

$$Y_t^\xi := \int_0^t \frac{1}{M_{s-}^\xi}\,\mathrm{d}M_s. \tag{11.7}$$

As $\mathrm{M}^\xi \in \mathscr{M}^2[0,T]$ (more precisely: $(\mathrm{M}^\xi)^T \in \mathscr{M}^2$) for all $T > 0$ and, by

$$\left|\frac{1}{M_{s-}^\xi}\right| = \left|e^{-t\psi(\xi)}\right| = e^{-t\mathrm{Re}\psi(\xi)} \leqslant 1,$$

also follows $\mathrm{Y}^\xi \in \mathscr{M}_0^2[0,T]$.

1° Y^ξ is an $\mathscr{F}_t$-LP($\mathbb{C}$) Fix $T > 0$, choose a partition $\pi = \left\{0 = t_0 < t_1 < \ldots < t_n = T\right\}$ and consider

$$\begin{aligned} Y^\pi(t) &= \sum_j \frac{1}{M_{t_{j-1}\wedge t}^\xi}\left(M_{t_j\wedge t}^\xi - M_{t_{j-1}\wedge t}^\xi\right) \\ &= \sum_j \left(e^{i\xi Z_{t_j\wedge t} - Z_{t_{j-1}\wedge t}} e^{-(t_j\wedge t - t_{j-1}\wedge t)\psi(\xi)} - 1\right). \end{aligned}$$

Clearly, since Z is an $\mathscr{F}_t$-LP(m),

$$Y^\pi(t_k) - Y^\pi(t_{k-1}) \underset{\text{suitable}}{\overset{\Phi}{=}} \Phi(Z_{t_k} - Z_{t_{k-1}}) \quad \perp\!\!\!\perp \quad \mathscr{F}_{t_{k-1}}.$$

Now, choose a sequence $\pi_n \subset \pi_{n+1} \subset \ldots$ of partitions with $|\pi_n| \to 0$ such that $\bigcup_n \pi_n$ is dense in $[0,T]$. Then

$$\begin{aligned} & Y^{\pi_n} \xrightarrow[n\to\infty]{\text{ucp}} Y^\xi && \text{(cf. §§ 8 and 10)} \\ \Longrightarrow\quad & Y_t - Y_s \perp\!\!\!\perp \mathscr{F}_s && s < t,\ s,t \in \bigcup_n \pi_n \end{aligned}$$

and by density and stochastic continuity it follows $Y_t - Y_s \perp\!\!\!\perp \mathscr{F}_s$ for all $s < t$. The conditions «stationary increments» and «càdlàg paths» are clear by ucp-convergence.

2° The MG $\mathrm{Y}^T \in \mathscr{M}_0^2$ has the unique decomposition

$$\mathrm{Y}^T = \mathrm{Y}^{T,c} + \mathrm{Y}^{T,d} \qquad \text{(cf. Definition 5.17).}$$

Hence, $\mathrm{Y}^{T,d} \in \mathscr{M}_0^{2,d}$ and for $t \leqslant T$

$$\Delta Y_t^T = \Delta Y_t^{T,d} = \frac{1}{M_{t-}^\xi}\Delta M_t^\xi = e^{i\xi\Delta Z_t} - 1.$$

Next,

$$\begin{aligned}
\mathbb{E}\left[Y^d, Y^d\right]_T &\overset{6.4\,(c)}{=} \mathbb{E}\sum_{t\leqslant T}\left|\Delta Y_t^T\right|^2\\
&= \mathbb{E}\sum_{t\leqslant T}\left|e^{i\xi\Delta Z_t}-1\right|^2\\
&= \mathbb{E}\int_0^T\int_{\mathbb{R}^m\setminus\{0\}}\left|e^{i\xi z}-1\right|^2 N(\mathrm{d}s,\mathrm{d}z)\\
&\overset{\text{BL}}{=} \lim_{\varepsilon\to 0}\mathbb{E}\int_0^T\int_{|z|>\varepsilon}\left|e^{i\xi z}-1\right|^2 N(\mathrm{d}s,\mathrm{d}z)\\
&\overset{8.14}{=} \lim_{\varepsilon\to 0}\mathbb{E}\int_0^T\int_{|z|>\varepsilon}\left|e^{i\xi z}-1\right|^2 \nu(\mathrm{d}z)\mathrm{d}s\\
&\overset{\text{BL}}{=} \mathbb{E}\int_0^T\int_{\mathbb{R}^m\setminus\{0\}}\left|e^{i\xi z}-1\right|^2 \nu(\mathrm{d}z)\mathrm{d}s\\
&= T\int_{\mathbb{R}^m\setminus\{0\}}\left|e^{i\xi z}-1\right|^2 \nu(\mathrm{d}z).
\end{aligned}$$

Because the LHS of the equality chain is finite, necessarily it follows that

$$\int_{\{|z|\leqslant 1\}}|z|^2\,\nu(\mathrm{d}z)<\infty \qquad\text{and}\qquad \int_{\{|z|>1\}}\nu(\mathrm{d}z)<\infty.$$

3° Define, for $t\leqslant T$,

$$\begin{aligned}
M'_t := \exp\Bigg(&Y_t^{\xi,c} + i\xi\left(\int_0^t\int_{|z|\leqslant 1} z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)+\int_0^t\int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z)\right)\\
&-\frac{1}{2}\left\langle Y^{\xi,c},Y^{\xi,c}\right\rangle_t + t\int_{0<|z|\leqslant 1}\left(1-e^{i\xi z}+i\xi z\right)\nu(\mathrm{d}z)+t\int_{|z|>1}\left(1-e^{i\xi z}\right)\nu(\mathrm{d}z)\Bigg).
\end{aligned}$$

Thus

$$M'_t = 1+\int_0^t M'_{s-}\,\mathrm{d}Y_s^{\xi}. \tag{11.8}$$

by applying Itô's formula (10.2) to M'_t for the function $F(x)=e^x$ – exercise! Since the solution to (11.8) is unique – as we will show later on, independently of the considerations here – it follows

$$M'_t = M_t^{\xi},\qquad t\leqslant T,$$

since, by (11.7), also

$$\mathrm{d}Y_s^{\xi} = \frac{1}{M_{s-}^{\xi}}\mathrm{d}M_s^{\xi} \qquad\Longrightarrow\qquad M_t^{\xi}\quad \text{solves (11.8) as well.}$$

4° Define

$$Y_t := \frac{1}{i\xi}Y_t^{\xi,c}+\int_0^t\int_{|z|\leqslant 1} z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)+\int_0^t\int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z).$$

Then it follows

$$e^{i\xi(Z_t-Y_t)} = M_t^{\xi} e^{-t\psi(\xi)} e^{-i\xi Y_t} \tag{11.9}$$
$$= e^{-t\psi(\xi)} \exp\left(-\frac{1}{2}\left\langle Y^{\xi,c}, Y^{\xi,c}\right\rangle_t + t\int \left(1-e^{i\xi z}+i\xi z\mathbb{1}_{\{|z|\leqslant 1\}}\right)\nu(\mathrm{d}z)\right).$$

By Theorem 10.4, $Y^{\xi,c} \perp\!\!\!\perp N(\mathrm{d}s,\mathrm{d}z)$ and Gaussian (ok, by 1^o, Theorem 10.4 can be adapted for $\mathbb{C}$-valued processes), i.e.

$$\left\langle Y^{\xi,c}, Y^{\xi,c}\right\rangle_t = -t\,\xi^2\sigma^2.$$

Therefore, $e^{i\xi(Z_t-Y_t)}$ and hence $Z_t - Y_t$ is deterministic and we can take the derivative wrt. t (the exponent is linear in t resp. the processes Y, Z are time-homogenous):

$$\frac{\mathrm{d}}{\mathrm{d}t}(Z_t - Y_t) = \lim_{s\to t}\frac{(Z_t-Y_t)-(Z_s-Y_s)}{t-s} = \lim_{h\to 0}\frac{Z_h - Y_h}{h} = \ell.$$

This shows

$$Y_t = Z_t - \ell t.$$

Since $Y_t^{\xi,c}$ is a centred $\mathbb{C}$-valued BM, so ℓ must be real and hence also Y_t is real.

$$\implies \qquad \frac{1}{i\xi}Y_t^{\xi,c} \quad \text{is a real BM.}$$

Finally, the Lévy-Khintchine formula eq. (11.6) follows from (11.9). ■

We want to mention a representation result for martingales. Therefore, let $(\mathscr{F}_t)_{t\geqslant 0}$ be a filtration generated by an $\mathscr{F}_t$-LP(m) Z $= (Z_t)_{t\geqslant 0}$.

Theorem 11.3 (Kunita; Watanabe). *Let* M $\in \mathscr{M}^2$ *an* $\mathscr{F}_t$*-MG. Then there are previsible processes* $f(s) = (f_1(s), \dots, f_m(s))$ *and* $g(s,z)$ *with*

$$\int_0^\infty |f(s)|^2\,\mathrm{d}s < \infty, \qquad \int_0^\infty\int_{z\neq 0} |g(s,z)|^2\,\nu(\mathrm{d}z)\mathrm{d}s < \infty, \tag{11.10}$$

and

$$M_t - M_0 = \sum_{j=1}^m \int_0^t f_j(s)\,\mathrm{d}B_s^j + \int_0^t\int_{z\neq 0} g(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z), \tag{11.11}$$

where $(B_s^1,\dots,B_s^m)$ *is a* BM(m) *and* N *the* $\mathscr{F}_t$*-PRM induced by* Z.

The «coefficients» f *and* g *are uniquely determined (up to indistinguishability) by* $(M_t)_{t\geqslant 0}$ *(i.e.* $\mathrm{d}s\otimes\mathbb{P}$*-a.s. and* $\mathrm{d}s\otimes\nu(\mathrm{d}z)\otimes\mathbb{P}$*-a.s. respectively).*

Proof. We refer the reader to the original work by H. Kunita [Kun04a].

The original proof goes back to H. Kunita and S. Watanabe in 1967 [KW67]. ■

Corollary 11.4. *Let* $Z_t, \mathscr{F}_t, B_t$ *and* $N(\mathrm{d}s,\mathrm{d}z)$ *as in the Theorem 11.3. Then we have*

$$\mathscr{M}_0^{2,c} := \left\{\sum_{j=1}^m\int_0^t f_j(s)\,\mathrm{d}B_s^j \,:\, \mathbb{E}\int_0^t |f(s)|^2\,\mathrm{d}s < \infty\right\}, \tag{11.12}$$

$$\mathscr{M}_0^{2,d} := \left\{\int_0^t\int_{\mathbb{R}^m\setminus\{0\}} g(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \,:\, g\in \mathrm{L}^2(\hat{N})\right\}. \tag{11.13}$$

Proof. Corollary 11.4 follows by Theorem 11.3 with the following formulae that are left as an exercise to the reader. Let M, N be martingales of the form

$$M_t = \sum_{j=1}^{m} \int_0^t f_j(s)\,\mathrm{d}B_s^j + \int_0^t \int_E g(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)$$

$$N_t = \sum_{j=1}^{m} \int_0^t \varphi_j(s)\,\mathrm{d}B_s^j + \int_0^t \int_E \gamma(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z).$$

Then we get

$$\langle M, N\rangle_t = \sum_{j=1}^{m} \int_0^t f_j(s)\varphi_j(s)\mathrm{d}s + \int_0^t \int_E g(s,z)\gamma(s,z)\nu(\mathrm{d}z)\mathrm{d}s$$

$$[M, N]_t = \sum_{j=1}^{m} \int_0^t f_j(s)\varphi_j(s)\mathrm{d}s + \int_0^t \int_E g(s,z)\gamma(s,z)N(\mathrm{d}s,\mathrm{d}z)$$

Note that, from Theorem 10.4 we know, it holds

$$\int_0^t f(s)\,\mathrm{d}B_s \quad \perp\!\!\!\perp \quad N_t(B) = \int_0^t \int_E \mathbb{1}_B(z)N(\mathrm{d}s,\mathrm{d}z).$$

This relation can be extended to $\widetilde{N}(\mathrm{d}s,\mathrm{d}z)$, to $g \in \mathscr{E}(\tilde{N})$, and finally by approximation to $\iint g(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)$. ■

Chapter 12

L^p ESTIMATES FOR STOCHASTIC INTEGRALS

We want to solve SDEs (*s*tochastic *d*ifferential *e*quations) and therefore we require estimates in L^p spaces as key tools. Typically, only the case $p = 2$ is considered, but $p > 2$ allows better regularity statements. Because those estimates are of general interest in themselves, we show also them here in the general context.

Without further mentioning, let

- $(\mathscr{F}_t)_{t\geqslant 0}$ be a fixed filtration;
- $(B_t)_{t\geqslant 0}$ be an $\mathscr{F}_t$-BM($\mathbb{R}^m$);
- $N(\mathrm{d}s, \mathrm{d}z)$ is an $\mathscr{F}_t$-PRM with compensator $\nu(\mathrm{d}z)\mathrm{d}s$;
- X_t an d-dimensional vector of SMG ($j = 1, 2, \dots, d$):

$$X_t^j = x_j + \int_0^t b_j(s)\mathrm{d}s + \sum_{k=1}^m \int_0^t f_{jk}(s)\,\mathrm{d}B_s^j + \int_0^t \int_E g_j(s,z)\widetilde{N}(\mathrm{d}s, \mathrm{d}z) \tag{12.1}$$

- $b = (b_1, \dots, b_d) \in \mathbb{R}^d$,

  $f = (f_{jk}) \in \mathbb{R}^{d\times m}, \quad 1 \leqslant j \leqslant d,\ 1 \leqslant k \leqslant m,$

  $g = (g_1, \dots, g_d) \in \mathbb{R}^d$

  $|f|^2 = \sum_{j,k} |f_{jk}|^2, \qquad |b|^2 = \sum_j |b_j|^2, \qquad |g|^2 = \sum_j |g_j|^2.$

Theorem 12.1. *For all $p \geqslant 2$ exists C_p with*

$$\begin{aligned}\mathbb{E}\sup_{0<s\leqslant t} |X_s|^p \leqslant C_p\Bigg\{ &\mathbb{E}|X_0|^p + \mathbb{E}\left[\int_0^t |b(r)|\,\mathrm{d}r\right]^p \\ &+ \mathbb{E}\left[\int_0^t |f(r)|^2\,\mathrm{d}r\right]^{p/2} \\ &+ \mathbb{E}\left[\int_0^t\int_E |g(r,z)|^2\,\nu(\mathrm{d}z)\mathrm{d}r\right]^{p/2} \\ &+ \mathbb{E}\int_0^t\int_E |g(r,z)|^p\,\nu(\mathrm{d}z)\mathrm{d}r\Bigg\}.\end{aligned} \tag{12.2}$$

Remark 12.2. For $\mathrm{X} \in \mathscr{M}_0^{2,c}$, i.e. $X_t = \int_0^t f(s)\,\mathrm{d}B_s$, (12.2) is one direction of the so called *Burkholder-Davis-Gundy* inequalities. Those equalities say that

$$\mathbb{E}\sup_{0<s\leqslant t}|X_s|^p \asymp \mathbb{E}\,\langle X, X\rangle_t^{p/2}, \qquad p > 0,\ t > 0.$$

with «comparing» constants being absolute and only depending of p, cf. e.g. [Pro05], p. 193 (Thm. 48 in Chapter IV.4), p. 222 (Thm. 73 in Chapter IV.7) or [IW89] (Second edition), p. 110 Thm. III.3.1.

Proof. Clearly,

$$\begin{aligned}\left|a_1+\ldots+a_d\right|^p \leqslant \left(\left|a_1\right|+\ldots+\left|a_d\right|\right)^p &\leqslant \left(d\cdot\max_{1\leqslant j\leqslant d}\left|a_j\right|\right)^p\\ &\leqslant d^p\left(\left|a_1\right|^p+\ldots+\left|a_d\right|^p\right)\\ &\leqslant d^{p+1}\left(\left|a_1\right|+\ldots+\left|a_d\right|\right)^p.\end{aligned}$$

Since the sup is subadditive, it suffices to estimate sup of the p-norm of each summand of (12.1) separately. The same holds true for the coordinate processes. WLOG let thus $d=1$ and WLOG let the RHS of (12.2) be finite.

Now, we estimate the summands in (12.1) separately.

$\boxed{1^o}$ $\mathbb{E}\left|X_0\right|^p$ is trivial.

$\boxed{2^o}$ We get

$$\mathbb{E}\left[\sup_{0<s\leqslant t}\left|\int_0^s b(r)\mathrm{d}r\right|^p\right]\leqslant\mathbb{E}\left[\sup_{0<s\leqslant t}\left(\int_0^s |b(r)|\,\mathrm{d}r\right)^p\right]=\mathbb{E}\left[\int_0^t |b(r)|\,\mathrm{d}r\right]^p.$$

$\boxed{3^o}$ We have $Y_t := \int_0^t \sum_{k=1}^m f_k(s)\,\mathrm{d}W_s^k \in \mathscr{M}_0^{2,c}[0,T]$ (if $p\geqslant 2$). Set

$$F(x) := |x|^p \quad\Longrightarrow\quad \begin{aligned}F'(x)&=px\,|x|^{p-1}\\ F''(x)&=p(p-1)\,|x|^{p-2}\end{aligned}$$

Now apply Itô's formula for continuous MG:

$$\left|Y_t\right|^p = p\int_0^t \left|Y_s\right|^{p-2}Y_s\,\mathrm{d}Y_s+\frac{1}{2}p(p-1)\int_0^t\left|Y_s\right|^{p-2}|f(s)|^2\,\mathrm{d}s.$$

Set $\tau_n := \inf\left\{t>0 : \left|Y_t\right|>n\right\}$. Then Y^{τ_n} is a bounded MG and by optional stopping

$$\begin{aligned}\mathbb{E}\left|Y_{t\wedge\tau_n}\right|^p &= \frac{p(p-1)}{2}\mathbb{E}\int_0^{t\wedge\tau_n}\left|Y_s\right|^{p-2}|f(s)|^2\,\mathrm{d}s\\ &\leqslant \frac{p(p-1)}{2}\mathbb{E}\left[\sup_{s\leqslant t\wedge\tau_n}\left|Y_s\right|^{p-2}\int_0^{t\wedge\tau_n}|f(s)|^2\,\mathrm{d}s\right]\\ &\overset{\text{Hölder}}{\underset{(\frac{p}{p-2},\frac{p}{2})}{=}} \frac{p(p-1)}{2}\mathbb{E}\left[\sup_{s\leqslant t\wedge\tau_n}\left|Y_s\right|^p\right]^{\frac{p-2}{2}}\left(\mathbb{E}\left[\int_0^{t\wedge\tau_n}|f(s)|^2\,\mathrm{d}s\right]^{\frac{p}{2}}\right)^{\frac{2}{p}}.\end{aligned}$$

We now apply Doob's maximal inequality with $1/p+1/q=1$:

$$\begin{aligned}\mathbb{E}\left[\sup_{s\leqslant t\wedge\tau_n}\left|Y_s\right|^p\right] &\overset{\text{Doob}}{\leqslant} q^p\mathbb{E}\left|Y_{t\wedge\tau_n}\right|^p\\ &\overset{(12.1)}{\leqslant}\frac{p(p-1)q^p}{2}\mathbb{E}\left[\sup_{s\leqslant t\wedge\tau_n}\left|Y_s\right|^p\right]^{\frac{p-2}{2}}\left(\mathbb{E}\left[\int_0^{t\wedge\tau_n}|f(s)|^2\,\mathrm{d}s\right]^{\frac{p}{2}}\right)^{\frac{2}{p}}.\end{aligned}$$

We can simplify this inequality by division of the second term and raising both sides by the power $p/2$. Then

$$\mathbb{E}\left[\sup_{s\leqslant t\wedge\tau_n}|Y_s|^p\right]\leqslant\left(\frac{p(p-1)q^p}{2}\right)^{\frac{p}{2}}\mathbb{E}\left[\int_0^{t\wedge\tau_n}|f(s)|^2\,ds\right]^{\frac{p}{2}}.$$

Taking $\tau_n\xrightarrow[n\to\infty]{\text{a.s.}}\infty$, the estimate of the third term in (12.2) follows with by means of Beppi Levi.

4° We now consider $Y_t:=\int_0^t\int_E g(s,z)\widetilde{N}(ds,dz)$. Applying Itô's formula (10.2) to $F(x)=|x|^p$, $p\geqslant 2$, we find

$$\begin{aligned}|Y_t|^p=&\underbrace{\int_0^t\int_E\left(|Y_{r-}+g(r,z)|^p-|Y_{r-}|^p\right)\widetilde{N}(ds,dz)}_{=:\,M_t}\\&+\int_0^t\int_E\left(|Y_{r-}+g(r,z)|^p-|Y_{r-}|^p-p\,|Y_{r-}|^{p-2}\,Y_{r-}\cdot g(s,z)\right)\nu(dz)ds.\end{aligned}\tag{12.3}$$

The process M_t is a local MG. Indeed: Because the compensator $\hat{N}(ds,dz)$ is continuous,

$$\sigma_n:=\inf\left\{t\geqslant 0:\mathbb{E}\int_0^t\int_E\left(|Y_{s-}+g(s,z)|^p+|Y_{s-}|^p\right)^2\hat{N}(ds,dz)\right\}\wedge n$$

is a reducing sequence for M_t, cf Lemma 9.4 (Exercise: Therefore note that integrability assumptions for ν and use Taylor's formula for the integrand).

Further we set

$$\tau_n:=\inf\left\{s\geqslant 0:|Y_s|>n\right\}\wedge\sigma_n.$$

By optional stopping, Y^{τ_n} is a bounded MG. By Taylor's formula – $g=g(r,z)$ –, we get

$$\begin{aligned}\left||Y_{s-}+g|^p+|Y_{s-}|^p-p\,|Y_{s-}|^{p-2}\,Y_{s-}\cdot g\right|&\overset{\text{Taylor}}{\underset{|\vartheta|\leqslant 1}{=}}\left|\frac{p(p-1)}{2}\,|Y_{s-}+\vartheta g|^{p-2}\,|g|^2\right|\\&\overset{(a+b)^u\leqslant 2^u(a^u+b^u)}{\leqslant}\frac{p(p-1)}{2}2^{p-2}\left(|Y_{s-}|^{p-2}\,|g|^2+|g|^p\right).\end{aligned}$$

Thus stopping in (12.3) by τ_n, we find taking expectations

$$\begin{aligned}\mathbb{E}\left|Y_{t\wedge\tau_n}\right|^p&\leqslant\frac{p(p-1)}{2}2^{p-2}\left\{\mathbb{E}\int_0^{t\wedge\tau_n}\int_E|Y_{s-}|^{p-2}\,|g|^2\,\nu(dz)ds+\mathbb{E}\int_0^{t\wedge\tau_n}\int_E|g|^p\,\nu(dz)ds\right\}\\&\leqslant c_p'\left\{\mathbb{E}\left[\sup_{s\leqslant t\wedge\tau_n}|Y_{s-}|^{p-2}\int_0^t\int_E|g|^2\,\nu(dz)ds\right]+\mathbb{E}\int_0^t\int_E|g|^p\,\nu(dz)ds\right\}\\&\overset{\text{Hölder}}{\underset{(\frac{p}{p-2},\frac{p}{2})}{\leqslant}}c_p'\left\{\left[\mathbb{E}\sup_{s\leqslant t\wedge\tau_n}|Y_{s-}|^p\right]^{\frac{p-2}{p}}\left[\mathbb{E}\left[\int_0^t\int_E|g|^2\,\nu(dz)ds\right]^{\frac{p}{2}}\right]^{\frac{2}{p}}+\mathbb{E}\int_0^t\int_E|g|^p\,\nu(dz)ds\right\}\\&\overset{\text{Young}}{=}c_p''\left\{\varepsilon^{\frac{p}{p-2}}\,\mathbb{E}\sup_{s\leqslant t\wedge\tau_n}|Y_{s-}|^p+\varepsilon^{-\frac{p}{2}}\mathbb{E}\left[\int_0^t\int_E|g|^2\,\nu(dz)ds\right]^{\frac{p}{2}}+\mathbb{E}\int_0^t\int_E|g|^p\,\nu(dz)ds\right\},\end{aligned}\tag{12.4}$$

where we used Young's inequality

$$AB = (\varepsilon A)(\varepsilon^{-1}B) \leqslant \varepsilon^{\alpha}\frac{A^{\alpha}}{\alpha} + \varepsilon^{-\beta}\frac{B^{\beta}}{\beta}, \qquad A, B \geqslant 0,\ \varepsilon > 0,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1$$

for $\alpha = p/(p-2)$ and $\beta = p/2$.

Finally, we apply the Doob maximal inequality on the MG Y^{τ_n}. Then we find

$$\begin{aligned}\mathbb{E}\sup_{s\leqslant t}\left|Y_{s\wedge\tau_n}\right| &\overset{\text{Doob}}{\leqslant} q^p\,\mathbb{E}\left|Y_{t\wedge\tau_n}\right|^p \\ &\overset{(12.4)}{\leqslant} q^p c_p'' \left\{\varepsilon^{\frac{p}{p-2}}\,\mathbb{E}\sup_{s\leqslant t\wedge\tau_n}\left|Y_{s-}\right|^p + \varepsilon^{-\frac{p}{2}}\mathbb{E}\left[\int_0^t\int_E |g|^2\,\nu(\mathrm{d}z)\mathrm{d}s\right]^{\frac{p}{2}} + \mathbb{E}\int_0^t\int_E |g|^p\,\nu(\mathrm{d}z)\mathrm{d}s\right\}.\end{aligned}$$

In particular, the LHS — also without the limit on the LHS!! — is finite. Choose $\varepsilon > 0$ so small that

$$q^p\,c_p''\,\varepsilon^{\frac{p}{p-2}} < \frac{1}{2}.$$

Hence,

$$\frac{1}{2}\mathbb{E}\sup_{s\leqslant t}\left|Y_{s\wedge\tau_n}\right| \leqslant q^p c_p'' \left\{\varepsilon^{-\frac{p}{2}}\mathbb{E}\left[\int_0^t\int_E |g|^2\,\nu(\mathrm{d}z)\mathrm{d}s\right]^{\frac{p}{2}} + \mathbb{E}\int_0^t\int_E |g|^p\,\nu(\mathrm{d}z)\mathrm{d}s\right\}.$$

As $\tau_n \xrightarrow[n\to\infty]{\text{a.s.}} \infty$ and since c_p'' does not depend on n, the fourth part of the estimate (12.2) follows again by monotone convergence. ■

Corollary 12.3. *Let X be a SMG of the form (12.1), $p \geqslant 2$ and $T > 0$. Then there are constants $C_{p,T}$ such that, for all $t \leqslant T$,*

$$\begin{aligned}\mathbb{E}\sup_{0<s\leqslant t}\left|X_s\right|^p \leqslant C_{p,T}\Bigg\{&\mathbb{E}\left|X_0\right|^p + \mathbb{E}\int_0^t |b(r)|^p\,\mathrm{d}r + \mathbb{E}\int_0^t |f(r)|^p\,\mathrm{d}r \\ &+ \mathbb{E}\int_0^t\left[\int_E |g(s,z)|^2\,\nu(\mathrm{d}z)\right]^{p/2}\mathrm{d}r + \mathbb{E}\int_0^t\int_E |g(r,z)|^p\,\nu(\mathrm{d}z)\mathrm{d}r\Bigg\}.\end{aligned} \tag{12.5}$$

Proof. The inequality (12.5) follows from (12.2) and Jensen's inequality. For example,

$$\begin{aligned}\left(\int_0^t |b(r)|\,\mathrm{d}r\right)^p &= t^p\left(\int_0^t |b(r)|\,\frac{\mathrm{d}r}{t}\right)^p \\ &\overset{\text{Jensen}}{\leqslant} t^p\int_0^t |b(r)|^p\,\frac{\mathrm{d}r}{t} \\ &\leqslant T^{p-1}\int_0^t |b(r)|^p\,\mathrm{d}r \qquad \forall t \leqslant T.\end{aligned}$$

■

We can even combine the estimate (12.5) with the Itô formula (10.2). Then we get

Theorem 12.4. *Let X be a SMG of the form (12.1), $p \geqslant 2$, $F \in C^2(\mathbb{R}^d)$ and $T > 0$. Then there are constants $C_{p,T}$ such that, for all $t \leqslant T$,*

$$
\begin{aligned}
\mathbb{E} \sup_{0<s\leqslant t} \left|F(X_s)\right|^p \leqslant C_{p,T} \Bigg\{ & \mathbb{E}\left|X_0\right|^p + \mathbb{E}\int_0^t \left|\nabla F(X_{r-})b(r)\right|^p \mathrm{d}r \\
& + \mathbb{E}\int_0^t \left(\sum_{k=1}^m \left|\nabla F(X_{r-}) f_{\bullet,k}(r)\right| \right)^{\frac{p}{2}} \mathrm{d}r \\
& + \frac{1}{2}\mathbb{E}\int_0^t \Bigg| \underbrace{\sum_{\ell=1}^m \sum_{j,k=1}^d \partial_j\, \partial_k F(X_{r-}) f_{j\ell}(r) f_{k\ell}(r)}_{\operatorname{tr}(F''\cdot f\cdot f^t)=\operatorname{tr}(f^t\cdot F''\cdot f)} \Bigg|^p \mathrm{d}r \\
& + \mathbb{E}\int_0^t \left(\int_E \left|F(X_{r-}+g(r,z)) - F(X_{r-})\right|^2 \nu(\mathrm{d}z) \right)^{\frac{p}{2}} \mathrm{d}r \\
& + \mathbb{E}\int_0^t \int_E \left|F(X_{r-}+g(r,z)) - F(X_{r-})\right|^p \nu(\mathrm{d}z)\mathrm{d}r \\
& + \mathbb{E}\int_0^t \left(\int_E \left|F(X_{r-}+g(r,z)) - F(X_{-r}) - \nabla F(X_{r-})g(r,z)\right| \nu(\mathrm{d}z) \right)^p \mathrm{d}r
\end{aligned}
\tag{12.6}
$$

Proof. DIY. ∎

Chapter 13

Stochastic Differential Equations

In this chapter we consider a class of SDEs (*s*tochastic *d*ifferential *e*quations) driven by relatively general SMG. Therefore let

- $(\mathscr{F}_t)_{t\geqslant 0}$ be a filtration;
- $B_t = (B_t^1, \dots, B_t^m)$ be an $\mathscr{F}_t$-adapted BM(m);
- $N(\mathrm{d}s, \mathrm{d}z)$ be an $\mathscr{F}_t$-PRM on $(0,\infty)\times E$;
- $b(\omega, t, x) = (b_1(\omega,t,x), \dots, b_d(\omega,t,x)) \in \mathbb{R}^d$,
  $f(\omega,t,x) = (f_{jk}(\omega,t,x))_{j,k} \in \mathbb{R}^{d\times m}$, $1\leqslant j\leqslant d$, $1\leqslant k\leqslant m$, and
  $g(\omega,t,x,z) = (g_1(\omega,t,x,z)), \dots, g_d(\omega,t,x,z)) \in \mathbb{R}^d$
  be previsible, i.e. $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)$ resp. $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}$ measurable processes with parameters $x\in\mathbb{R}^d$ and $z\in E$.

Moreover, we consider the following parameter dependent SMG

$$X_j(t,x) = \int_0^t b_j(r,x)\mathrm{d}r + \sum_{k=1}^m \int_0^t f_{jk}(r,x)\,\mathrm{d}B_r^k + \int_0^t\int_E g_j(r,x,z)\widetilde{N}(\mathrm{d}r,\mathrm{d}z), \tag{13.1}$$

which can also be written equivalently using the vector notation

$$X(t,x) = \int_0^t b(r,x)\mathrm{d}r + \underbrace{\int_0^t f(r,x)\,\mathrm{d}B_r}_{\int \text{matrix}\times\text{vector}} + \int_0^t\int_E g(r,x,z)\widetilde{N}(\mathrm{d}r,\mathrm{d}z). \tag{13.2}$$

By means of those notations we can define SDEs. Therefore, let $(\eta_t)_{t\geqslant 0}$ an d-dimensional adapted càdlàg process. Then $b(t,\eta_{t-})$, $f(t,\eta_{t-})$, $g(t,\eta_{t-},z)$ are previsible[1] and we **define**

$$\begin{aligned}\int_{t_0}^t X(\mathrm{d}r,\eta_{r-}) := \int_{t_0}^t b(r,\eta_{r-})\mathrm{d}r + \int_{t_0}^t f(r,\eta_{r-})\,\mathrm{d}B_r \\ + \int_{t_0}^t\int_E g(r,\eta_{r-},z)\widetilde{N}(\mathrm{d}r,\mathrm{d}z).\end{aligned} \tag{13.3}$$

[1]Indeed: As compositions of measurable processes

$$b(\omega,t,\eta_{t-}(\omega)) = \underbrace{b(\omega,t,\bullet)}_{\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R}^d)} \circ \underbrace{\eta_{t-}(\omega)}_{\mathscr{P}/\mathscr{B}(\mathbb{R}^d)}.$$

Definition 13.1. Let $\xi_0 \in \mathscr{F}_{t_0}$ be a RV. An $\mathscr{F}_t$-adapted càdlàg process $\xi_t = (\xi_t^1, \ldots, \xi_t^d) \in \mathbb{R}^d$ is called **solution to the SDE**

$$d\xi_t = X(dt, \xi_{t-}), \qquad \xi_{t_0} = \xi_0 \tag{13.4}$$

with initial condition ξ_0 if it holds

$$\xi_t = \xi_0 + \int_{t_0}^t X(dr, \xi_{r-}). \tag{13.5}$$

Further, we call

- $X(t, x)$ the **generator** of the solution ξ_t;
- $b(t, x)$ the **drift coefficient**;
- $f(t, x)$ the **diffusion coefficient**;
- $g(t, x, z)$ the **jump coefficient**.

If we want to prove existence (and uniqueness) of solutions to the SDE (13.5) we require two key assumptions on the coefficients appearing in (13.1) or (13.2) respectively.[2]

Linear Growth Conditions There are constants $C, C(z)$ such that

$$\left.\begin{aligned} |b(\omega,t,x)| &\leqslant C\,(1+|x|) && \mathbb{P}\text{-a.s.} \quad \forall t,x;\\ |f(\omega,t,x)| &\leqslant C\,(1+|x|) && \mathbb{P}\text{-a.s.} \quad \forall t,x;\\ |g(\omega,t,x,z)| &\leqslant C(z)\,(1+|x|) && \mathbb{P}\text{-a.s.} \quad \forall t,x,z;\\ \int_E |C(z)|^p\,\nu(dz) &< \infty && \text{for } p=2 \text{ and some } p \geqslant 2. \end{aligned}\right] \tag{LG}$$

Lipschitz Conditions Let $K \subset \mathbb{R}^d$. Then there are constants $L, L(z)$ such that

$$\left.\begin{aligned} |b(\omega,t,x)-b(\omega,t,y)| &\leqslant L\,|x-y| && \mathbb{P}\text{-a.s.} \quad \forall t,x,y;\\ |f(\omega,t,x)-f(\omega,t,y)| &\leqslant L\,|x-y| && \mathbb{P}\text{-a.s.} \quad \forall t,x,y;\\ |g(\omega,t,x,z)-g(\omega,t,y,z)| &\leqslant L(z)\,|x-y| && \mathbb{P}\text{-a.s.} \quad \forall t,x,y,z;\\ \int_E |L(z)|^p\,\nu(dz) &< \infty && \text{for } p=2 \text{ and some } p \geqslant 2. \end{aligned}\right] \tag{Lip}$$

Mind Clearly, it follows

$$(\text{Lip}) \quad \text{for } K = \mathbb{R}^d \qquad \Longrightarrow \qquad (\text{LG}).$$

Theorem 13.2 (Existence & Uniqueness). *Let $p \geqslant 2$ and the coefficients b, f and g satisfy (LG) and (Lip) for $K = \mathbb{R}^d$. If $\mathbb{E}\left|\xi_0\right|^p < \infty$, then the SDE (13.4) has eine unique solution $(\xi_t)_{t\geqslant 0} \subset L^p(\mathbb{P})$ with initial condition $\xi_{t_0} = \xi_0$.*
Additionally *We even get $\sup_{t_0<s\leqslant t}\left|\xi_s\right| \in L^p(\mathbb{P})$ for all $t \geqslant t_0$.*

[2] Note that $|\cdot|$ always denotes the ℓ^2-norm.

Proof. We use the classical Picard iteration for L^p. Therefore, let $\xi_0 \in L^p(\mathscr{F}_{t_0}, \mathbb{P})$. We set recursively ($n \in \mathbb{N}$)

$$\begin{aligned} \xi_t^0 &:= \xi_0 \\ \xi_t^n &:= \xi_0 + \int_{t_0}^t X(\mathrm{d}s, \xi_{s-}^{n-1}). \end{aligned} \tag{13.6}$$

Then,

$$\xi_t^1 = \xi_0 + \int_{t_0}^t b(r, \xi_0)\mathrm{d}r + \int_{t_0}^t f(r, \xi_0)\,\mathrm{d}B_r + \int_{t_0}^t \int_E g(r, \xi_0, z)\widetilde{N}(\mathrm{d}r, \mathrm{d}z).$$

Clearly ξ_t^1 is a càdlàg SMG. By Theorem 12.1, we see

$$\begin{aligned} \mathbb{E}\left[\sup_{t_0 \leqslant r \leqslant t} \left|\xi_r^1 - \xi_r^0\right|^p\right] \leqslant \Bigg\{ &\mathbb{E}\left[\int_{t_0}^t \left|b(r, \xi_0)\right| \mathrm{d}r\right]^p \\ &+ \mathbb{E}\left[\int_{t_0}^t \left|f(r, \xi_0)\right|^2 \mathrm{d}r\right]^{\frac{p}{2}} \\ &+ \mathbb{E}\left[\int_0^t \int_E \left|g(r, \xi_0, z)\right|^2 \nu(\mathrm{d}z)\mathrm{d}r\right]^{\frac{p}{2}} \\ &+ \mathbb{E}\int_0^t \int_E \left|g(r, \xi_0, z)\right|^p \nu(\mathrm{d}z)\mathrm{d}r \Bigg\} \\ \overset{\text{(LG)}}{\leqslant} &\; C_p' \,\mathbb{E}\left[1 + \left|\xi_0\right|^p\right]. \end{aligned}$$

Hence, $(\xi_t^1)_{t \geqslant 0}$ is an L^p-bounded SMG.

Analogously we see that, if $(\xi_t^{n-1})_{t \geqslant 0}$ is an L^p-MG, then

$$\begin{aligned} \mathbb{E}\left[\sup_{t_0 \leqslant r \leqslant t} \left|\xi_r^n - \xi_r^{n-1}\right|^p\right] &\leqslant C_p' \,\mathbb{E}\left[1 + \left|\xi_t^{n-1}\right|^p\right] \\ &\leqslant C_{p,n}' \,\mathbb{E}\left[1 + \left|\xi_0\right|^p\right], \end{aligned}$$

i.e. all $(\xi_t^n)_{t \geqslant 0}$ are L^p-bounded SMG.

Next, we use the Lipschitz conditions (Lip)

$$\begin{aligned} \mathbb{E}\left[\sup_{t_0 \leqslant r \leqslant t} \left|\xi_r^{n+1} - \xi_r^n\right|^p\right] &\leqslant \mathbb{E}\left[\sup_{t_0 \leqslant s \leqslant t} \left|\int_{t_0}^s X(\mathrm{d}r, \xi_{r-}^n) - \int_{t_0}^s X(\mathrm{d}r, \xi_{r-}^{n-1})\right|^p\right] \\ &\leqslant \mathbb{E}\Bigg[\sup_{t_0 \leqslant r \leqslant t} \bigg|\int_{t_0}^s \left(b(r, \xi_{r-}^n) - b(r, \xi_{r-}^{n-1})\right) \mathrm{d}r + \int_{t_0}^s \left(f(r, \xi_{r-}^n) - f(r, \xi_{r-}^{n-1})\right) \mathrm{d}B_r \\ &\qquad + \int_{t_0}^s \int_E \left(g(r, \xi_{r-}^n, z) - g(r, \xi_{r-}^{n-1}, z)\right) \widetilde{N}(\mathrm{d}r, \mathrm{d}z)\bigg|^p\Bigg] \\ &\overset{12.1}{\leqslant} c_p \Bigg\{ \mathbb{E}\left[\int_{t_0}^s \left|b(r, \xi_{r-}^n) - b(r, \xi_{r-}^{n-1})\right| \mathrm{d}r\right]^p \\ &\qquad + \mathbb{E}\left[\int_{t_0}^t \left|f(r, \xi_{r-}^n) - f(r, \xi_{r-}^{n-1})\right|^2 \mathrm{d}r\right]^{\frac{p}{2}} \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\left[\int_{t_0}^{t}\int_E \left|g(r,\xi^n_{r-},z)-g(r,\xi^{n-1}_{r-},z)\right|^2 \nu(\mathrm{d}z)\mathrm{d}r\right]^{\frac{p}{2}}\\
&+\mathbb{E}\int_{t}^{t_0}\int_E \left|g(r,\xi^n_{r-},z)-g(r,\xi^{n-1}_{r-},z)\right|^p \nu(\mathrm{d}z)\mathrm{d}r\Bigg\}\\
&\overset{\text{(Lip)}}{\leqslant} c_p'\int_{t_0}^{t}\mathbb{E}\left|\xi^n_{r-}-\xi^{n-1}_{r-}\right|^p\mathrm{d}r\\
&\leqslant c_p'\int_{t_0}^{t}\mathbb{E}\left[\sup_{t_0\leqslant r\leqslant s}\left|\xi^n_{r-}-\xi^{n-1}_{r-}\right|^p\right]\mathrm{d}r.
\end{aligned}
$$

Thus we have an inequality of the form

$$\sigma_{n+1}(t)\leqslant c\int_{t_0}^{t}\sigma_n(t_n)\mathrm{d}t_n.$$

By iteration, we get

$$
\begin{aligned}
\sigma_{n+1}(t)&\leqslant c^2\int_{t_0}^{t}\int_{t_0}^{t_n}\sigma_{n-1}(t_{n-1})\mathrm{d}t_{n-1}\mathrm{d}t_n\\
&\ \vdots\\
&\leqslant c^n\int_{t_0}^{t}\int_{t_0}^{t_n}\dots\int_{t_0}^{t_2}\sigma_1(t_1)\mathrm{d}t_1\dots\mathrm{d}t_n\\
&\leqslant c^n\underbrace{\sup_{t_0\leqslant s\leqslant t}\sigma_1(s)}_{\leqslant\sigma_1(t)}\underbrace{\int_{t_0}^{t}\int_{t_0}^{t_n}\dots\int_{t_0}^{t_2}\sigma_1(t_1)\mathrm{d}t_1\dots\mathrm{d}t_n}_{=(t-t_0)^n/n!\leqslant t^n/n!}.
\end{aligned}
$$

■

[**Geometric interpretation of the integral expression** Assume all integral upper bounds would be $= t$. Then all integrals would be just the volume of the n-dimensional hypercube with side length $t - t_0$. Because in our case we have «triangle integrals», we only consider a part of the hypercube lying below its main diagonal. By symmetry that must be $n!$ parts, i.e. the integral expression is exactly $(t - t_0)^n/n!$. However, this can be also verified by a direct calculation.]

Consequently,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t_0\leqslant r\leqslant t}\left|\xi^{n+1}_r-\xi^n_r\right|^p\right]&\leqslant\frac{c_p^nt^n}{n!}\mathbb{E}\left[\sup_{t_0\leqslant r\leqslant t}\left|\xi^1_r-\xi^0_r\right|^p\right]\\
&\leqslant\frac{c_p^nt^n}{n!}\mathbb{E}\left[1+\left|\xi_0\right|^p\right].
\end{aligned}
$$

By the Monkowski inequality for norms, it follows for $m < n$

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t_0\leqslant r\leqslant t}\left|\xi^n_r-\xi^m_r\right|^p\right]^{\frac{1}{p}}&\leqslant\sum_{j=m}^{n+1}\mathbb{E}\left[\sup_{t_0\leqslant r\leqslant t}\left|\xi^{j+1}_r-\xi^j_r\right|^p\right]^{\frac{1}{p}}\\
&\leqslant\sum_{j=m}^{\infty}\left[\frac{c_p^nt^n}{n!}\right]^{\frac{1}{p}}\mathbb{E}\left[1+\left|\xi_0\right|^p\right]\xrightarrow{m\to\infty}0.
\end{aligned}
$$

Hence, $\xi^n_t\xrightarrow[\mathrm{L}^p]{\text{loc. uniform in } t}\xi_t$ converges and thus it is clear that $(\xi_t)_{t\geqslant0}$ is adapted, càdlàg, and a solution to (13.3).

Finally, we show *uniqueness*. Therefore, let $(\xi_t)_{t\geqslant 0}$ and $(\xi'_t)_{t\geqslant 0}$ be two solution in L^p. Then

$$\xi_t - \xi'_t = \int_{t_0}^{t} X(\mathrm{d}r, \xi_{r-}) - \int_{t_0}^{t} X(\mathrm{d}r, \xi'_{r-})$$

and the same inequalities as above show that

$$\mathbb{E}\left[\sup_{t_0\leqslant r\leqslant t} |\xi_r - \xi'_r|^p\right] \leqslant c\int_{t_0}^{t} \mathbb{E}\left[\sup_{t_0\leqslant r\leqslant s} |\xi_r - \xi'_r|^p\right] \mathrm{d}s.$$

This is (again) an inequality of the type

$$\varphi(t) \leqslant c\int_{t_0}^{t} \varphi(s)\mathrm{d}s,$$

so by iteration of this inequality we get

$$\begin{aligned}\varphi(t) &\leqslant c^n \int_{t_0}^{t}\int_{t_0}^{t_n}\dots\int_{t_0}^{t_2} \varphi(t_1)\mathrm{d}t_1\dots\mathrm{d}t_n\\ &\leqslant \frac{c^n t^n}{n!}\varphi(t) \xrightarrow{n\to\infty} 0.\end{aligned}$$

Hence $\varphi = 0$, i.e. ξ and ξ' are indistinguishable on all intervals $[t_0, n]$, $n \in \mathbb{N}$.

In fact, in the situation of Theorem 13.2, we can get much strong existence statements. Namely,

Theorem 13.3. *Under the assumptions of Theorem 13.2 every solution $(\xi_t)_{t\geqslant 0}$ with initial condition $\xi_{t-} = \xi_0 \in \mathrm{L}^p(\mathscr{F}_{t_0}, \mathbb{P})$ is L^p-bounded. In particular, in this situation the SDE (13.4) is always admits a unique solution.*

Proof. By assumption

$$\xi^1_t = \xi_{t_0} + \int_{t_0}^{t} b(r, \xi_{r-})\mathrm{d}r + \int_{t_0}^{t} f(r, \xi_{r-})\,\mathrm{d}B_s + \int_{t_0}^{t}\int_E g(r, \xi_{r-}, z)\widetilde{N}(\mathrm{d}r, \mathrm{d}z).$$

We define a sequence of ST

$$\tau_n := \left\{t \geqslant t_0 : \left|\xi_t\right| > n\right\}.$$

Since $t \mapsto \xi_t$ is càdlàg, then $\tau_n \xrightarrow[n\to\infty]{\text{a.s.}} \infty$. By Theorem 12.1 for $\xi_t^{\tau_n}$

$$\begin{aligned}\mathbb{E}\left[\sup_{t_0\leqslant r\leqslant t\wedge\tau_n} |\xi_t^{\tau_n}|^p\right] \leqslant C_p\Bigg\{&\mathbb{E}\left|\xi_{t_0}\right|^p\\ &+\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left|b(r,\xi_{r-})\right|\mathrm{d}r\right]^p\\ &+\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left|f(r,\xi_{r-})\right|^2\mathrm{d}r\right]^{p/2}\\ &+\mathbb{E}\left[\int_{t_0}^{t\wedge\tau_n}\int_E \left|g(r,\xi_{r-}z)\right|^2 \nu(\mathrm{d}z)\mathrm{d}r\right]^{p/2}\\ &+\mathbb{E}\int_{t_0}^{t\wedge\tau_n}\int_E \left|g(r,\xi_{r-}z)\right|^p \nu(\mathrm{d}z)\mathrm{d}r\Bigg\}\end{aligned}$$

$$\leqslant C_p' \left\{ \mathbb{E}\left|\xi_{t_0}\right|^p + \mathbb{E}\left[1 + \int_{t_0}^{t\wedge\tau_n} |\xi_{r-}|^p \, \mathrm{d}r\right] \right\}.$$

$$\leqslant C_{p,t} \left\{ 1 + \int_{t_0}^{t} \underbrace{\mathbb{E}\left[\sup_{t_0 \leqslant s \leqslant r\wedge\tau_n} |\xi_{r-}|^p\right]}_{\leqslant n \text{ since } r\leqslant\tau_n} \mathrm{d}r \right\}.$$

which is an inequality of the type

$$\begin{aligned}
\varphi(t) &\leqslant ca + c\int_{t_0}^{t} \varphi(t_n)\mathrm{d}t_n \\
&\leqslant ca + \int_{t_0}^{t} c^2 a\mathrm{d}t_n + \int_{t_0}^{t_n} c^2\varphi(t_{n-1})\mathrm{d}t_{n-1}\mathrm{d}t_n \\
&\ \vdots \\
&\leqslant ca + c^2at + c^3a^2t^2/2! + \ldots + c^{n+1}a^nt^n/n! \\
&\quad + \underbrace{\int_{t_0}^{t}\int_{t_0}^{t_n}\cdots\int_{t_0}^{t_2} c^{n+1}\varphi(t_1)\mathrm{d}t_1\ldots\mathrm{d}t_n}_{\xrightarrow{n\to\infty} 0} \\
&\leqslant c\left(a + (e^{cat} - 1)\right).
\end{aligned}$$

Consequently for $c = c(p,t)$

$$\mathbb{E}\left[\sup_{t_0\leqslant r\leqslant t\wedge\tau_n} |\xi_t|^p\right] \leqslant c\left(a + (e^{cat} - 1)\right) = ce^{ct}.$$

The RHS does not depend of n. Therefore, we can apply Beppo Levi for taking the limit $n \to \infty$ and find

$$\mathbb{E}\left[\sup_{t_0\leqslant r\leqslant t} |\xi_r|^p\right] \leqslant c(p,t)\, e^{c(p,t)t} \qquad \forall t \geqslant t_0. \qquad \blacksquare$$

Finally, we want to localise the Lipschitz condition (Lip): Therefore, let $K = \overline{B(0,n)}$ and $\chi_n \in \mathrm{C}_c(\mathbb{R}^d)$ with $\mathbb{1}_{B(0,2n)} \leqslant \chi \leqslant \mathbb{1}_{B(0,3n)}$. We set the coefficients as

$$b_n(\omega,t,x) := b(\omega,t,x)\chi_n(x) \qquad \text{and} \qquad f_n := f\chi_n, \quad g_n := g\chi_n.$$

Clearly, for any $n \in \mathbb{N}$,

$$b, f, g \quad \text{satisfy eq. (Lip) on } B(0,n) \qquad \Longrightarrow \qquad b_n, f_n, g_n \quad \text{satisfy eq. (Lip) on } \mathbb{R}^d.$$

Caution The Lipschitz constants depends on n: $L = L_n$ and $L(z) = L_n(z)$!

Proposition 13.4. *Let (LG) and (Lip) be satisfied on B(0,n) for any $n \in \mathbb{N}$. Then the SDE (13.4) has a unique solution.*

Proof. We set

$$\xi_t^N := \xi_0 + \int_{t_0}^{t} X^N(\mathrm{d}r, \xi_{r-}^N). \tag{13.7}$$

Here let X^N defined as X in (13.2) but with coefficients b_N, f_N, g_N. Using that there is a unique solution to the SDE (13.7) (cf. Theorem 13.3) thus

$$\xi_t^M = \xi_t^M \qquad \forall t < \tau_M, \forall M \leqslant N,$$

where

$$\tau_M := \left\{t > t_0 : \xi_t^M > M\right\}.$$

Then, by setting

$$\xi_t := \xi_t^M \qquad \forall t < \tau_M,$$

ξ_t is a solution to the original SDE (13.4), since by the localisation principle for SDEs, it holds

$$\int_{t_0}^{t\wedge\tau_M} X^M(\mathrm{d}r, \xi_{r-}^M) = \int_{t_0}^{t\wedge\tau_M} X(\mathrm{d}r, \xi_{r-}).$$

The solution is unique, as it is unique on all intervals (t_0, τ_M). ■

Remark 13.5. In the literature (e.g. [Pro05]) one often finds SDEs of the following form:

$$\begin{aligned}\mathrm{d}\xi_t &= \vec{v}(\xi_{t-})\mathrm{d}\vec{Z}_t \\ &= \sum_{j=1}^{m} v_j(\xi_{t-})\mathrm{d}Z_t^j\end{aligned} \tag{13.8}$$

with a general SMG $\vec{Z} = (Z^1, \dots, Z^m)$. Such SDEs are called **separated** and they are less general than the SDEs considered by us – modulo a BV part in the driven integrator. Indeed: with the generator

$$X(t,x) := \sum_{j=1}^{m} v_j(x)Z_t^j \tag{13.9}$$

we can write (13.9) as (13.5).

The Gronwall inequality

The Gronwall lemma is a famous result from the theory of ordinary differential equations. Therein, it is often used to show uniqueness results. Indirectly, we have already used this lemma in the previous proofs for the stochastic differential equations (iteration trick) and even proved, respectively.

Theorem 13.6 (Gronwall). *Let $u, a, b : [0,\infty) \to [0,\infty)$ positive, measurable functions satisfying the following differential equations:*

$$u(t) \leqslant a(t) + \int_0^t b(s)u(s)\mathrm{d}s. \tag{13.10}$$

Then

$$u(t) \leqslant a(t) + \int_0^t a(s)u(s)\exp\left(\int_s^t b(r)\mathrm{d}r\right)\mathrm{d}s. \tag{13.11}$$

Proof. Let $y(t) := \int_0^t b(s)u(s)\mathrm{d}s$. Since y is the primitive of $b(t)u(t)$, we can write (13.10) for (Lebesgue) almost all t also as

$$y'(t) - b(t)y(t) \leqslant a(t)b(t). \tag{13.12}$$

We define $z(t) := y(t)\exp\left(-\int_0^t b(s)\mathrm{d}s\right)$ and put this term into formula (13.12). Then it follows

$$z'(t) \leqslant a(t)b(t)\exp\left(-\int_0^t b(s)\mathrm{d}s\right)$$

Lebesgue almost everywhere. On the other hand, $z(0) = y(0) = 0$ and we get by integrating

$$z(t) \leqslant \int_0^t a(s)b(s)\exp\left(-\int_0^s b(r)\mathrm{d}r\right)\mathrm{d}s,$$

so

$$y(t) \leqslant \int_0^t a(s)b(s)\exp\left(\int_s^t b(r)\mathrm{d}r\right)\mathrm{d}s$$

for all $t > 0$. Consequently, (13.6), since $u(t) \leqslant a(t) + y(t)$. ■

Chapter 14

Dependence of an SDE on the Initial Conditions

In this chapter we study the dependence of solutions to SDEs from its staring point. As in § 13 therefore let

$$X(t,x)=\int_0^t b(s,x)\mathrm{d}s+\int_0^t f(s,x)\,\mathrm{d}B_s+\int_0^t\int_E g(s,x,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)$$

and

$$\xi_t=\xi_0+\int_{t_0}^t X(\mathrm{d}s,\xi_{s-}),\qquad \xi_0=x\in\mathbb{R}^d. \tag{14.1}$$

Here again: $b=(b_1,\dots,b_d)\in\mathbb{R}^d$, $f=(f_{k\ell})_{k,\ell=1}^{d,m}\in\mathbb{R}^{d\times m}$, $g=(g_1,\dots,g_d)\in\mathbb{R}^d$ and $B=(B_t^1,\dots,B_t^m)$ an is an m-dimensional BM. Moreover, let the coefficients b,f,g satisfy the linear growth (LG) and Lipschitz conditions (Lip). Without proof we cite Kolmogorov's continuity criterion:

Theorem 14.1 (Kolmogorov; Chentsov; Totoki). *(a) Let $(X(x))_{x\in D}$ be a process with values in $\mathbb{R}^n$ $\big[$ or more general: in a normed space $(\mathsf{B},|\cdot|)\big]$ and parameter set $D\subset\mathbb{R}^d$. If there are constants $\alpha,\beta,C>0$ with*

$$\mathbb{E}\,|X(x)-X(y)|^\alpha\leqslant C\,|x-y|^{\beta+d}\qquad\forall\,|x-y|\leqslant 1, \tag{14.2}$$

then $(X(x))_{x\in D}$ has a continuous *modification.*

(b) (Totoki) The modification is uniformly continuous on D and we can thus extend $X(x)$ uniquely on $\overline{D}$ by continuity, i.e. there is a uniquely determined, continuous process $\big(\tilde{X}(x)\big)_{x\in\overline{D}}$ such that $\tilde{X}\big|_D=X$.

Proof. A proof can be found e.g. in

- [Bau02] §§ 39.3-39.5 (Kolmogorov-Chentsov, $d=1$),
- [Kal21] Theorem 3.23 (Kolmogorov-Chentsov, $d\in\mathbb{N}$),
- [Kun04b] Appendix (Kolmogorov-Chentsov-Totoki, $d\in\mathbb{N}$). ■

Our goal is to prove inequalities of the type (14.2) for the SDE considered by us.

Theorem 14.2 (Path cty. dependence on the initial value). *Under the assumptions of Theorem 13.2 (i.e. (LG) & (Lip)) we find for the solution $\xi_t=\xi_t(x)$ of (14.1) with initial value $\xi_0(x)=x$ that*

$$\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\big(1+|\xi_s(x)|\big)^p\right]\leqslant c_p\left(1+|x|^p\right)\qquad\forall t\leqslant t, \tag{14.3}$$

$$\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}|\xi_s(x)-\xi_s(y)|^p\right]\leqslant c_p'\,|x-y|^p\qquad\forall t\leqslant t. \tag{14.4}$$

Proof. (14.3) follows almost in the same way as the proof of (13.7) and is extremely similar to the estimate (14.4). Thus we will only show the estimate (14.4).

Set $\Delta_t := \xi_t(x) - \xi_t(y)$. Then

$$\begin{aligned}\Delta_t = (x-y) &+ \int_{t_0}^t \big(b(r,\xi_{r-}(x)) - b(r,\xi_{r-}(y))\big)\,\mathrm{d}r \\ &+ \int_{t_0}^t \big(f(r,\xi_{r-}(x)) - f(r,\xi_{r-}(y))\big)\,\mathrm{d}B_r \\ &+ \int_{t_0}^t \int_E \big(g(r,\xi_{r-}(x),z) - g(r,\xi_{r-}(y),z)\big)\,\widetilde{N}(\mathrm{d}r,\mathrm{d}z).\end{aligned}$$

We now apply Corollary 12.3 — i.e. the L^p estimate, based on Theorem 12.1 —, then

$$\begin{aligned}\mathbb{E}\sup_{t_0\leqslant s\leqslant t}\left|\Delta_t\right|^p &\leqslant c_{p,t}\Big\{\,|x-y|^p \\ &\qquad + \mathbb{E}\int_{t_0}^t \left|b(r,\xi_{r-}(x)) - b(r,\xi_{r-}(y))\right|^p \mathrm{d}r \\ &\qquad + \mathbb{E}\int_{t_0}^t \left|f(r,\xi_{r-}(x)) - f(r,\xi_{r-}(y))\right|^p \mathrm{d}r \\ &\qquad + \mathbb{E}\int_{t_0}^t \left(\int_E \left|g(r,\xi_{r-}(x),z) - g(r,\xi_{r-}(y),z)\right|^2 \nu(\mathrm{d}z)\right)^{p/2} \mathrm{d}r \\ &\qquad + \mathbb{E}\int_{t_0}^t \int_E \left|g(r,\xi_{r-}(x),z) - g(r,\xi_{r-}(y),z)\right|^p \nu(\mathrm{d}z)\mathrm{d}r\Big\} \\ &\overset{\text{(Lip)}}{\leqslant} c'_{p,t}\Big\{\,|x-y|^p \\ &\qquad + \left(2L^p + \left[\int_E L^2(z)\nu(\mathrm{d}z)\right]^{p/2} + \int_E L^p(z)\nu(\mathrm{d}z)\right)\int_{t_0}^t \mathbb{E}\left|\xi_{r-}(x) - \xi_{r-}(y)\right|^p \mathrm{d}r \\ &\leqslant c''_{p,t}\left\{\,|x-y|^p + \int_{t_0}^t \mathbb{E}\left[\sup_{t_0\leqslant s\leqslant r}\left|\Delta_s\right|^p\right]\mathrm{d}r\right\}.\end{aligned}$$

As in the proof of Theorem 13.2 we get from this inequality

$$\begin{aligned}\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\Delta_s\right|^p\right] &\leqslant c\left[|x-y|^p + \left(e^{ct|x-y|^p} - 1\right)\right] \\ &\leqslant \tilde{c}_{p,t}\,|x-y|^p,\end{aligned}$$

where the last estimate follows from the elementary convexity inequality

$$e^u - 1 \leqslant (e-1)u \qquad \forall u \in (0,1).$$

Finally, we apply Theorem 14.1 with the parameters

$$\begin{gathered}\alpha = \beta + d \equiv p, \\ X(x) = \left\{\xi_s(x) \,:\, s \in [t_0,t]\right\}, \\ (\mathsf{B}, |\cdot|) = \mathsf{C}_b([t_0,t]; \|\cdot\|_\infty).\end{gathered}$$

■

Similarly to continuity, we can prove differentiability of the solution with respect to the initial value. Let us briefly sketch the proof strategy:

$$e_j = (\underbrace{0, \dots, 0, 1}_{j=1,\dots,d}, 0, \dots, 0) \in \mathbb{R}^d.$$

Further, we set

$$\frac{1}{h}\Delta_{he_j}\xi_t(x) := \frac{1}{h}\left(\xi_t(x+he_j) - \xi_t(x)\right).$$

From the proof of Theorem 14.2, we know that $(x, h) \mapsto h^{-1}\Delta_{he_j}\xi_t(x)$ is continuous on $\mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$.

Indeed: If $\exists \lim_{h\to\infty} h^{-1}\Delta_{he_j}\xi_t(x)$, then $\xi_t(x)$ is partially differentiable in the direction of e_j.

Next, we show that $h^{-1}\Delta_{he_j}\xi_t(x)$ is also uniformly continuous in $(x, h) \in \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$. Thus we can apply Kolmogorov-Chentsov-Totoki (Theorem 14.1) and find[1]

$$\lim_{h\to 0}\frac{1}{h}\Delta_{he_j}\xi_t(x) = \underbrace{\nabla_j\xi_t(x) \in \mathbb{R}^d}_{\text{cts. in } x}.$$

Then $\xi_t(x)$ is continuously (totally) differentiable because all partial derivatives do exist and are continuous.

We require also additional differentiability conditions on the coefficients on the SDE. Namely,

$$\begin{aligned}
\exists \qquad & \nabla b := (\nabla_1 b, \dots, \nabla_d b) = (\partial_j b_k)_{j,k=1,\dots,d}, \\
\exists \qquad & \nabla f := (\nabla_1 f, \dots, \nabla_d f) = (\partial_j f_{k\ell})_{\substack{j,k=1,\dots,d, \\ \ell=1,\dots,m}}, \\
\exists \qquad & \nabla g := (\nabla_1 g, \dots, \nabla_d g) = (\partial_j g_k)_{j,k=1,\dots,d}, \\
& |\nabla b| + |\nabla f| \leqslant K', \\
& |\nabla g| \leqslant K'(z), \\
& \int_E K'(z)^q \nu(\mathrm{d}z) < \infty \qquad \forall q \in [2, p].
\end{aligned} \tag{14.5}$$

Moreover, we assume for some $\varepsilon \in (0, 1]$

$$\begin{aligned}
|\nabla b(t,x) - \nabla b(t,y)| &\leqslant L'\,|x-y|^\varepsilon \\
|\nabla f(t,x) - \nabla f(t,y)| &\leqslant L'\,|x-y|^\varepsilon \\
|\nabla g(t,x,z) - \nabla g(t,y,z)| &\leqslant L'(z)\,|x-y|^\varepsilon \\
\int_E L'(z)^q \nu(\mathrm{d}z) &< \infty \qquad \forall q \in [2, p].
\end{aligned} \tag{14.6}$$

Note the eqs. (14.5) and (14.6) holds $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$-almost surely.

[1] Here $\nabla_j f := (\partial_j f^1, \dots, \partial_j f^d)$.

Theorem 14.3. *Assume the coefficients of the SDE (14.1) satisfy the assumption eqs. (14.5), (14.6), (LG) and (Lip). Then, for $p \geqslant 2$, there is a constant C_p such that on $D := \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$*

$$\mathbb{E}\left[\sup_{t_0 \leqslant s \leqslant t} \left|\frac{1}{h}\Delta_{he_j}\xi_s(x)\right|^p\right] \leqslant C_p \qquad \forall (h,x) \in D, \tag{14.7}$$

$$\mathbb{E}\left[\sup_{t_0 \leqslant s \leqslant t} \left|\frac{1}{h}\Delta_{he_j}\xi_s(x) + \frac{1}{h}\Delta_{h'e_j}\xi_s(x')\right|^p\right] \leqslant C_p\left(\left|x-x'\right|^{\varepsilon p} + \left|h-h'\right|^{\varepsilon p}\right) \tag{14.8}$$

$$\forall (h,x),(h',x') \in D,\ \left|x-x'\right| + \left|h-h'\right| \leqslant 1.$$

Proof. Set

$$\begin{aligned}\Delta_t' := \frac{1}{h}\Delta_{he_j}\xi_t(x) = e_j &+ \frac{1}{h}\int_{t_0}^t \Delta_{he_j} b(r,\xi_{r-}(x))\mathrm{d}r \\ &+ \frac{1}{h}\int_{t_0}^t \Delta_{he_j} f(r,\xi_{r-}(x))\,\mathrm{d}B_r \\ &+ \frac{1}{h}\int_{t_0}^t \int_E \Delta_{he_j} g(r,\xi_{r-}(x),z)\widetilde{N}(\mathrm{d}r,\mathrm{d}z).\end{aligned}$$

Here we use the convention that

$$\Delta_{he_j}\Phi(r,\xi_{r-}(x),\dots) := \Phi(r,\xi_{r-}(x+he_j),\dots) - \Phi(r,\xi_{r-}(x),\dots).$$

We want to apply Corollary 12.3 again. First, let us consider the term $h^{-1}\Delta_{he_j}b(r,\xi_{r-}(x))$.

We therefore recall the multidimensional Taylor formula, or mean value theorem, respectively: If $\Psi \in \mathrm{C}^1(\mathbb{R}^d,\mathbb{R}^m)$. Then

$$\Psi(\xi+\eta) - \Psi(\xi) = \int_0^1 \underbrace{\nabla\Psi}_{\in\mathbb{R}^{m\times d}}(\xi + \vartheta\eta)\mathrm{d}\vartheta \cdot \eta \in \mathbb{R}^m.$$

So it follows

$$\begin{aligned}\frac{1}{h}\Delta_{he_j}b(r,\xi_{r-}(x)) &= \frac{1}{h}\left[b(r,\xi_{r-}(x+he_j)) - b(r,\xi_{r-}(x))\right] \\ &= \frac{1}{h}\left[b(r,\xi_{r-}(x) + \Delta_{he_j}\xi_{r-}(x)) - b(r,\xi_{r-}(x))\right] \\ &= \frac{1}{h}\int_0^1 \nabla b(r,\xi_{r-}(x) + \vartheta\Delta_{he_j}\xi_{r-}(x))\mathrm{d}\vartheta \cdot \Delta_{he_j}\xi_{r-}(x).\end{aligned}$$

However, by assumption $|\nabla b| \leqslant K'$. Thus

$$\mathbb{E}\int_{t_0}^t \left|\frac{1}{h}\Delta_{he_j}b(r,\xi_{r-}(x))\right|^p \mathrm{d}r \leqslant K''\mathbb{E}\int_{t_0}^t \left|\frac{1}{h}\Delta_{he_j}\xi_{r-}(x)\right|^p \mathrm{d}r.$$

And analogous we see that

$$\mathbb{E}\int_{t_0}^t \left|\frac{1}{h}\Delta_{he_j}f(r,\xi_{r-}(x))\right|^p \mathrm{d}r \leqslant K''\mathbb{E}\int_{t_0}^t \left|\frac{1}{h}\Delta_{he_j}\xi_{r-}(x)\right|^p \mathrm{d}r.$$

By

$$\frac{1}{h}\Delta_{he_j}g(r,\xi_{r-}(x),z) = \frac{1}{h}\int_0^1 \nabla g(r,\xi_{r-}(x) + \vartheta\Delta_{he_j}\xi_{r-}(x),z)\mathrm{d}\vartheta \cdot \Delta_{he_j}\xi_{r-}(x)$$

it follows, by the boundedness of the coefficients (14.5)

$$\mathbb{E}\int_{t_0}^{t}\left(\int_E\left|\frac{1}{h}\Delta_{he_j}g(r,\xi_{r-}(x),z)\right|^2\nu(\mathrm{d}z)\right)^{p/2}\mathrm{d}r+\mathbb{E}\int_{t_0}^{t}\int_E\left|\frac{1}{h}\Delta_{he_j}g(r,\xi_{r-}(x),z)\right|^p\nu(\mathrm{d}z)\mathrm{d}r$$
$$\leqslant K'''\mathbb{E}\int_{t_0}^{t}\left|\frac{1}{h}\Delta_{he_j}\xi_{r-}(x)\right|^p\mathrm{d}r.$$

Altogether we get, by Corollary 12.3,

$$\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\frac{1}{h}\Delta_{he_j}\xi_s(x)\right|^p\right]=C\left(1+\int_{t_0}^{t}\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant r}\left|\frac{1}{h}\Delta_{he_j}\xi_s(x)\right|^p\right]\mathrm{d}r\right)$$

from which by the usual methods (cf. Theorem 14.2, Gronwall etc.) follows the inequality (14.7).

Finally, to prove the inequality (14.8), we consider

$$\begin{aligned}&\frac{1}{h}\Delta_{he_j}\xi_t(x)+\frac{1}{h'}\Delta_{h'e_j}\xi_t(x')\\&\quad=\int_{t_0}^{t}\left[\frac{1}{h}\Delta_{he_j}b(r,\xi_{r-}(x))+\frac{1}{h'}\Delta_{h'e_j}b(r,\xi_{r-}(x'))\right]\mathrm{d}r\\&\qquad+\int_{t}^{t_0}\left[\frac{1}{h}\Delta_{he_j}f(r,\xi_{r-}(x))+\frac{1}{h'}\Delta_{h'e_j}f(r,\xi_{r-}(x'))\right]\mathrm{d}B_r\\&\qquad+\int_{t}^{t_0}\int_E\left[\frac{1}{h}\Delta_{he_j}g(r,\xi_{r-}(x),z)+\frac{1}{h'}\Delta_{h'e_j}g(r,\xi_{r-}(x'),z)\right]\widetilde{N}(\mathrm{d}r,\mathrm{d}z).\end{aligned}$$

Further, by the mean value theorem (or Taylor expansion)

$$\begin{aligned}&\frac{1}{h}\Delta_{he_j}b(r,\xi_{r-}(x))+\frac{1}{h'}\Delta_{h'e_j}b(r,\xi_{r-}(x'))\\&\leqslant\int_0^1\nabla b(r,\xi_{r-}(x)+\vartheta\Delta_{he_j}b(r,\xi_{r-}(x))\mathrm{d}\vartheta\cdot\frac{1}{h}\Delta_{he_j}\xi_{r-}(x)\\&\quad-\int_0^1\nabla b(r,\xi_{r-}(x')+\vartheta\Delta_{h'e_j}b(r,\xi_{r-}(x'))\mathrm{d}\vartheta\cdot\frac{1}{h}\Delta_{h'e_j}\xi_{r-}(x')\\&\leqslant\int_0^1\left[\nabla b(r,\xi_{r-}(x)+\vartheta\Delta_{he_j}b(r,\xi_{r-}(x))-\nabla b(r,\xi_{r-}(x')+\vartheta\Delta_{h'e_j}b(r,\xi_{r-}(x'))\right]\mathrm{d}\vartheta\cdot\frac{1}{h}\Delta_{he_j}\xi_{r-}(x)\\&\quad+\int_0^1\nabla b(r,\xi_{r-}(x')+\vartheta\Delta_{h'e_j}b(r,\xi_{r-}(x'))\mathrm{d}\vartheta\cdot\left(\frac{1}{h}\Delta_{he_j}\xi_{r-}(x)-\frac{1}{h'}\Delta_{h'e_j}\xi_{r-}(x')\right)\\&\overset{(14.5)}{\underset{(14.6)}{\leqslant}}\left(L'\left|\xi_{r-}(x)-\xi_{r-}(x')\right|^{\varepsilon}+L'\left|\Delta_{he_j}\xi_{r-}(x)-\Delta_{h'e_j}\xi_{r-}(x')\right|^{\varepsilon}\right)\frac{1}{h}\left|\Delta_{he_j}\xi_{r-}(x)\right|\\&\quad+K'\left|\frac{1}{h}\Delta_{he_j}\xi_{r-}(x)-\frac{1}{h'}\Delta_{h'e_j}\xi_{r-}(x')\right|.\end{aligned}$$

Using $\mathbb{E}\,|Z|\leqslant\sqrt{\mathbb{E}Z^2}$, we find

$$\begin{aligned}&\mathbb{E}\int_{t_0}^{t}\left|\frac{1}{h}\Delta_{he_j}b(r,\xi_{r-}(x))-\frac{1}{h'}\Delta_{h'e_j}b(r,\xi_{r-}(x'))\right|^p\mathrm{d}r\\&\quad\leqslant C_1\int_{t_0}^{t}\left(\mathbb{E}\left|\xi_{r-}(x)-\xi_{r-}(x')\right|^{2\varepsilon p}\right)^{1/2}\mathrm{d}r\\&\qquad+C_2\int_{t_0}^{t}\left(\mathbb{E}\left|\xi_{r-}(x+he_j)-\xi_{r-}(x'+h'e_j)\right|^{2\varepsilon p}\right)^{1/2}\mathrm{d}r\end{aligned}$$

$$
\begin{aligned}
&+ C_3 \int_{t_0}^{t} \mathbb{E}\left|\frac{1}{h}\Delta_{he_j}\xi_s(x) + \frac{1}{h'}\Delta_{h'e_j}\xi_s(x')\right|^p \mathrm{d}r \\
&\overset{14.2}{\leqslant} C_1'\left|x-x'\right|^{\varepsilon p} + C_2'\left|h-h'\right|^{\varepsilon p} + C_3 \int_{t_0}^{t} \mathbb{E}\left|\frac{1}{h}\Delta_{he_j}\xi_{r-}(x) + \frac{1}{h'}\Delta_{h'e_j}\xi_{r-}(x')\right|^p \mathrm{d}r.
\end{aligned}
$$

Analogous we find for the diffusion coefficient

$$
\begin{aligned}
&\mathbb{E}\int_{t_0}^{t}\left|\frac{1}{h}\Delta_{he_j} f(r,\xi_{r-}(x)) - \frac{1}{h'}\Delta_{h'e_j} f(r,\xi_{r-}(x'))\right|^p \mathrm{d}r \\
&\quad \overset{14.2}{\leqslant} C_1'\left|x-x'\right|^{\varepsilon p} + C_2'\left|h-h'\right|^{\varepsilon p} + C_3 \int_{t_0}^{t} \mathbb{E}\left|\frac{1}{h}\Delta_{he_j}\xi_s(x) + \frac{1}{h'}\Delta_{h'e_j}\xi_s(x')\right|^p \mathrm{d}r.
\end{aligned}
$$

For the jump coefficient g, finally

$$
\begin{aligned}
&\left|\frac{1}{h}\Delta_{he_j} g(r,\xi_{r-}(x),z) - \frac{1}{h'}\Delta_{h'e_j} g(r,\xi_{r-}(x'),z)\right| \\
&\quad \leqslant L'(z)\left(\left|\xi_{r-}(x)-\xi_{r-}(x')\right|^{\varepsilon} + \left|\xi_{r-}(x+he_j)-\xi_{r-}(x'+h'e_j)\right|^{\varepsilon}\right)\cdot \frac{1}{h}\Delta_{he_j}\xi_{r-}(x) \\
&\qquad + K'(z)\left|\frac{1}{h}\Delta_{he_j}\xi_{r-}(x) + \frac{1}{h'}\Delta_{h'e_j}\xi_{r-}(x')\right|.
\end{aligned}
$$

Thus we find

$$
\begin{aligned}
&\mathbb{E}\int_{t_0}^{t}\left(\int_E \left|\frac{1}{h}\Delta_{he_j} g(r,\xi_{r-}(x),z) - \frac{1}{h'}\Delta_{h'e_j} g(r,\xi_{r-}(x'),z)\right| \nu(\mathrm{d}z)\right)^{p/2} \mathrm{d}r \\
&\quad \leqslant C_4\left(\left|x-x'\right|^{\varepsilon p} + \left|h-h'\right|^{\varepsilon p}\right) + C_5 \int_{t_0}^{t} \mathbb{E}\left|\frac{1}{h}\Delta_{he_j}\xi_{r-}(x) + \frac{1}{h'}\Delta_{h'e_j}\xi_{r-}(x')\right|^p \mathrm{d}r,
\end{aligned}
$$

and also

$$
\begin{aligned}
&\mathbb{E}\int_{t_0}^{t}\int_E \left|\frac{1}{h}\Delta_{he_j} g(r,\xi_{r-}(x),z) - \frac{1}{h'}\Delta_{h'e_j} g(r,\xi_{r-}(x'),z)\right|^p \nu(\mathrm{d}z)\mathrm{d}r \\
&\quad \leqslant C_4\left(\left|x-x'\right|^{\varepsilon p} + \left|h-h'\right|^{\varepsilon p}\right) + C_5 \int_{t_0}^{t} \mathbb{E}\left|\frac{1}{h}\Delta_{he_j}\xi_{r-}(x) + \frac{1}{h'}\Delta_{h'e_j}\xi_{r-}(x')\right|^p \mathrm{d}r.
\end{aligned}
$$

Thus we have estimated all four terms appearing in the L^p estimates of Corollary 12.3.

Altogether, we get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\frac{1}{h}\Delta_{he_j}\xi_s(x) - \frac{1}{h'}\Delta_{h'e_j}\xi_s(x')\right|^p\right] \\
&\quad \leqslant C\left(\left|x-x'\right|^{\varepsilon p} + \left|h-h'\right|^{\varepsilon p} + \int_{t_0}^{t}\mathbb{E}\left|\frac{1}{h}\Delta_{he_j}\xi_r(x) + \frac{1}{h'}\Delta_{h'e_j}\xi_r(x')\right|^p \mathrm{d}r\right).
\end{aligned}
$$

Finally, by Grownwall's inequality (or as in the proof of Theorem 14.2), we see that

$$
\mathbb{E}\left[\sup_{t_0\leqslant s\leqslant t}\left|\frac{1}{h}\Delta_{he_j}\xi_s(x) - \frac{1}{h'}\Delta_{h'e_j}\xi_s(x')\right|^p\right] \leqslant C_{p,t}\left(\left|x-x'\right|^{\varepsilon p} + \left|h-h'\right|^{\varepsilon p}\right),
$$

for all $x, x' \in \mathbb{R}^d$ and $h, h' > 0$ with $\left|x-x'\right| + \left|h-h'\right| \leqslant 1$. Hence the claim follows. ■

Corollary 14.4. *Under the assumptions of Theorem 14.3 the solution $\xi_t(x)$ of the SDE (14.1) is almost surely differentiable in x for all $t \geqslant t_0$.*

The derivative $\nabla\xi_t(x)$ satisfies the (formally differentiable) SDE

$$\nabla\xi_t(x) = \mathrm{id}_{\mathbb{R}^d} + \int_{t_0}^{t} \nabla X(\mathrm{d}r, \xi_{r-}(x)) \cdot \nabla\xi_{r-}(x) \tag{14.9}$$

with generator $\nabla\ X(t,x) \in \mathbb{R}^{d\times d}$ given by

$$\begin{aligned}\nabla X(t,x) = \int_0^t \nabla b(x,r)\mathrm{d}r + \sum_{\ell=1}^{m} \int_0^t \overbrace{\underbrace{\nabla f_{\bullet,\ell}(x,r)}_{\mathbb{R}^d \ni \ell th\ column}}^{\in\mathbb{R}^{d\times d}\ \forall \ell}\ \mathrm{d}\ \overbrace{B}^{\mathbb{R}\ni\ell}{}^{\ell}_{r} \\ + \int_0^t \int_E \nabla g(r,x,z)\widetilde{N}(\mathrm{d}r, \mathrm{d}z).\end{aligned} \tag{14.10}$$

Proof. We only sketch the proof.

1° The (continuous) differentiability follows by Theorem 14.3 and the previously made remarks.

2° We remark that by the chain rule

$$\begin{aligned}\frac{1}{h}\Delta_{he_j} b(r, \xi_{r-}(x)) &= \frac{b(r, \xi_{r-}(x+he_j)) - b(r, \xi_{r-}(x))}{h} \\ &\xrightarrow{h\to\infty} \Big(\underbrace{\nabla b(r, \xi_{r-}(x))}_{\in\mathbb{R}^{d\times d}} \cdot \underbrace{\nabla\xi_{r-}(x)}_{\in\mathbb{R}^{d\times d}}\Big)e_j \in \mathbb{R}^d.\end{aligned}$$

The corresponding convergence necessary holds for $g \in \mathbb{R}^d$ and every column $f_{\bullet,\ell} \in \mathbb{R}^d$ of the diffusion matrix f.

3° Altogether it then follows that $\nabla\xi_t(x)$ (uniquely) solves the (formally differentiated) SDE (14.10). ■

* * * END * * *

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Subject index